Reply to "Multifractality: at least three moments!" by Schertzer et al.

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We thank very much D. Schertzer, I. Tchiguirinskaia, and S. Lovejoy for their thoughtful contribution (hereinafter Schertzer et al., 2013) to the discussion following the publication of our manuscript (Lombardo et al., 2013) in HESSD. We address below four main points. To improve readability, we quoted the original comments as indented Italic text.

1. *First, the conclusions of this paper, as well as part of its title, are misleading. The authors indeed claim that only the first and second statistical moments of a multifractal field are safely estimated. Unfortunately, these two moments are insufficient to determine the nonlinear scaling moment function K(q) of a multifractal field R, e.g. the rain rate, observed at various resolutions $\lambda = L/l$ (where L is the largest scale, l is the observation scale) and where < . > denotes a given ensemble average, $<R_\lambda>\approx\lambda^{K(q)}$ (1). Indeed, contrary to the linear case of a scaling moment function K(q) of a uni/monofractal field, a third independent value is at least required to estimate the curvature of K(q) for multifractal fields. Therefore, their conclusions would mean that the multifractal parameters could not be safely assessed and multifractals would be therefore of little interest to simulate rainfall.*

We thank the Commenters for calling our attention on this important issue, which we will clarify in the revised version of our manuscript. Indeed, the questions (a) how many moments we can estimate reliably and (b) how many moments are needed for a certain model (e.g. multifractal) are two different questions. We focus only on the first question in our work (Lombardo et al., 2013). Nonetheless, if we can reliably estimate just two moments, as we illustrate in the paper, and a model needs three, then we agree with the Schertzer et al. (2013) that there is a problem. Moreover, in the literature, we may see that many authors do not estimate/use three moments, but they go up to 6-7th order (or even more) to justify the use of their multifractal model. Here stands our critique: we cannot use unreliably calculated quantities to justify our models.

Based on our findings, which are consistent with previous literature results (see e.g. Ossiander and Waymire, 2000), we conclude that (especially in case of temporal dependence and subexponential distribution tails) it is crucial to investigate uncertainty in moment estimation, even when using low-order moments only, i.e. up to the second order. For example, in the paper Fig. 2, we show that, for highly correlated data series, the estimates of the first-order moment are of much lower quality than the iid case. Besides, Koutsoyiannis and Montanari (2007) demonstrated that the second moment is again very uncertain in case of dependence. Consequently, the moment of order three is bound to be caught in a worse fate (as shown in our paper Figs. 3, 5, 6 and 7). However, in the conclusions of Lombardo et al. (2013) we stated that "Moments of order > 3 should be avoided in model identification and fitting because their estimation is problematic", hence we do not exclude the third moment. We only suggest not to use it if we cannot be aware of its uncertainty (which is commonly the case in statistical hydrology).

Thus, our conclusions would mean that if we have to use higher order moments – e.g. the third raw moment to determine the nonlinear scaling moment function $K(q)$ in the assessment of a multifractal behaviour – it is imperative to specify their uncertainty and involve it in modelling and inference.

Nevertheless, we agree with Schertzer et al. (2013) that we may have distracted the reader from the message we wanted to convey. Indeed, as stated in the abstract by Lombardo et al. (2013): "In particular, we suggest to use the first two moments in all problems as they suffice to define the most important characteristics of the distribution". The problem is in "suffice". For multifractal models they do not suffice, particularly, in justifying or rejecting them.

In the revised version of our manuscript, we will rephrase the above statement as follows: "In particular, we suggest that, because of estimation problems, the use of moments of order higher than two should be avoided, either in justifying or fitting models. Nonetheless, in most problems the first two moments provide enough information for the most important characteristics of the distribution".

2. *Secondly, the present paper fully ignores the concept of second order multifractal phase transition, introduced years ago (...). This phase transition explains not only in a straightforward manner the qualitative observations made by the authors that the estimates of the statistical moments of order $q \geq 3$ of their numerically simulated cascades seem to be spurious, but provides rigorous, analytical results. Indeed, this phase transition occurs at a critical moment order q_s that is analytically defined from $K(q)$ and such that the estimates over a sample of all the statistical moments of order $q > q_s$ are spurious. We show below that the discrete cascade model (Lombardo et al., 2012) used by the authors yields a theoretical critical order $q_s(k)$ that depends on the k^{th} level of the cascade. Its graph is displayed in Fig.1 for the parameter set chosen by the authors, but with a much larger number of steps to show its slow convergence to $q_s(\infty) \approx 2.582$. Because this asymptotic value is also an upper bound, the estimates of the moments of order $q > q_s(\infty)$ are all spurious. The simulations and*

qualitative observations of the present paper can be therefore seen as illustrations of this phase transition rather than new findings.

We again thank the Commenters for this interesting criticism that allows us to clarify the framework into which our analyses and results are embedded. To accomplish this aim, we emphasize that Lombardo et al. (2013) use two different approaches in Monte Carlo experiments. Both of them are formulated in a way to generate stationary time series exhibiting two main characteristics observed in many geophysical processes (such as rainfall), i.e. the long-term persistence (ruled by the Hurst coefficient, see below) and subexponential tails of the marginal distribution (except for the Gaussian distribution which is used as benchmark, because it has been dominating in classical statistical applications). Therefore, we do not necessarily aim at simulating multifractal processes. In any case, we appreciate that Schertzer et al. (2013) demonstrate that our results are consistent with those based on the concept of second-order phase transition in the multifractal framework (although we prefer statistical arguments and calculations over metaphorically introduced concepts like that of critical order).

Going into detail of the approaches used in our Monte Carlo experiments (see also the paper Sect. 3.2.1), we should stress that in the first approach we simulate lognormal time series by a downscaling model (Lombardo et al., 2012), which is characterized by a simple cascade structure. The synthetic series are generated by developing the cascade from a large scale to our scale of interest. In our reply to the Reviewer Pierluigi Furcolo, we prove that our model exhibits a multifractal behaviour asymptotically. Hence, the scaling of raw moments cannot be extended to the entire range of scales considered in our analysis. The reader is referred to the reply to Furcolo for further details on this point.

In the second approach, we first generate time series at the finest scale and then aggregate them into coarser scales. We simulate synthetic series following Pareto, lognormal, Weibull (with shape parameter smaller than 1) and Gaussian distributions in order to investigate the influence of the distribution tails on the prediction intervals of sample moments (see e.g. the paper Fig. 7).

Lombardo et al. (2013) show that both approaches described above (which use different models) lead to the same conclusions.

Finally, we highlight that our results provide insight into the capability of raw moments of different orders in supporting reliable inferences about a natural behaviour or in fitting of models. This is accomplished through a quantitative evaluation of the efficiency of raw moment estimators.

3. *The (Hurst) scaling exponent $H = -K(1)$ of the mean field. The value $H = 0$ corresponds to a "conservative field", i.e. a field whose mean is strictly scale invariant, whereas $H \neq 0$ rather corresponds to a fractional integration/derivation of a conservative field.*

We believe a clarification about what we mean by Hurst exponent (or Hurst coefficient) *H* may be useful to avoid misunderstanding. According to the definition given by Mandelbrot and Van Ness (1968): "By 'fractional Brownian motions' (fBm's), we propose to designate a family of Gaussian random functions defined as follows: $\underline{B}(t)$ being ordinary Brownian motion, and H a parameter satisfying $0<H<1$, fBm of exponent H [denoted as $\underline{B}_H(t)$] is a moving average of $d\underline{B}(t)$, in which past increments of $\underline{B}(t)$ are weighted by the kernel $(t-s)^{H-1/2}$". As usual, t designates time, $-\infty<t<\infty$. The increment process, $\underline{x}(t_2-t_1)=\underline{B}_H(t_2)-\underline{B}_H(t_1)$, is known as fractional Gaussian noise (i.e. Hurst-Kolmogorov process) and it is given by (see Mandelbrot and Van Ness, 1968, p. 424):

$$\underline{x}(t_2-t_1)=\frac{1}{\Gamma(H+1/2)}\left(\int_{-\infty}^{t_2}(t_2-s)^{H-1/2}\,d\underline{B}(s)-\int_{-\infty}^{t_1}(t_1-s)^{H-1/2}\,d\underline{B}(s)\right) \quad (1)$$

which is a fractional integral in the sense of Weyl.

The reason for introducing the stationary stochastic process described above is that the "span of interdependence" between its random variables can be said to be infinite, thus resembling the strong interdependence between distant samples observed in many empirical studies in diverse fields of science, such as hydrology (e.g. Koutsoyiannis, 2013). Indeed, the original motivation in introducing fBm (as stated by Mandelbrot and Van Ness, 1968) came from Hurst's pioneering findings when studying the records of water flows through the Nile river (Hurst, 1951). Therefore, the exponent *H* of the fBm was named after Hurst.

The value of the Hurst coefficient *H* determines three very different families of fBm's, corresponding, respectively, to: $0<H<1/2$, $1/2<H<1$, and $H=1/2$.

For our paper purposes, we restrict ourselves to a discussion of Hurst-Kolmogorov processes positively correlated, i.e. $1/2<H<1$. These processes are all characterized by long-term persistence which is associated with power-law correlations, and often referred to as Hurst effect (Koutsoyiannis, 2002).

To conclude, we wish to emphasize that fractal behaviour and long-term persistence of a process are, in general, different things. The former is characterized by the fractal dimension, a measure of roughness, while the latter is identified by the Hurst coefficient *H*, a measure of long-range dependence. As stated by Gneiting and Schlather (2004): "In principle, fractal dimension and Hurst coefficient are independent of each other: fractal dimension is a local property, and long-memory dependence is a global characteristic". Some multifractal analyses usually confuse the two.

For further details, the reader is also pointed to our Comment entitled: "Is consistency a limitation? – Reply to "Further (monofractal) limitations of climactograms" by Lovejoy et al.", by Koutsoyiannis et al. (2013).

4. *To apply these ideas to the present model* [Lombardo et al., 2012] *the only difficulty is to go through the cumbersome algebra of its scale dependent parameters, which implies a rather unusual scale dependence of the scaling moment function K(q, k).*

The moment scaling function $K(q)$ represents the asymptotic slope of the raw moments as the scale tends to zero (Falconer, 1990). In the reply to the Reviewer Pierluigi Furcolo, we prove that the fact that multifractal behaviour is variable should not be unusual but rather common in our opinion. According to our calculations, indeed, any stationary and ergodic model has necessarily a variable behaviour. It cannot be otherwise, because for scales tending to infinity the $K(q)$ should tend to zero, while for scales tending to zero the $K(q)$ will take nonzero values.

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