

# Bilinear surface smoothing for spatial interpolation with optional incorporation of an explanatory variable.

## Part 1: Theory

Nikolaos Malamos<sup>1\*</sup> and Demetris Koutsoyiannis<sup>2</sup>

<sup>1</sup>*Department of Agricultural Technology, Technological Educational Institute of Western Greece, Amaliada, Greece*

<sup>2</sup>*Department of Water Resources and Environmental Engineering, School of Civil Engineering, National Technical University of Athens, Zographou, Greece*

\*[nmalamos@teimes.gr](mailto:nmalamos@teimes.gr)

**Abstract** Bilinear surface smoothing is an alternative concept which provides flexible means for spatial interpolation. Interpolation is accomplished by means of fitting a bilinear surface into a regression model with known break points and adjustable smoothing terms. Additionally, as an option, the incorporation in an objective manner, of the influence of an explanatory variable available at a considerable denser dataset is possible. The parameters involved in each case (with or without an explanatory variable) are determined by a nonparametric approach based on the generalized cross-validation (GCV) methodology. A convenient search technique of the smoothing parameters was achieved by transforming them in terms of tension parameters, with values restricted in the interval [0, 1). The mathematical framework, the computational implementation and details concerning both versions of the methodology, as well as practical aspects of their application are presented and discussed. In a companion paper, examples using both synthesized and real world (hydrological) data are presented to illustrate the methodology. The proposed mathematical framework constitutes a simple alternative to existing spatial interpolation methodologies.

**Key words** bilinear surface smoothing; spatial interpolation; explanatory variable; generalized cross-validation; linear smoothers; LU decomposition

## INTRODUCTION

In multidimensional interpolation, we seek estimates of the dependent variables at points placed inside the analysis space that forms regular or irregular sized grids. In order to achieve such an objective, various techniques have been deployed; many of them can be applied to perform spatial interpolation of environmental variables that are usually collected from point measurements.

These methodologies fall into three main categories ([Li and Heap 2008](#)):

- (1) non-geostatistical methods such as: Splines, Thin Plate Splines ([Craven and Wahba 1979](#), [Wahba and Wendelberger 1980](#)) and Regression Methods ([Davis 1986](#));
- (2) geostatistical methods including different approaches of Kriging, such as: Ordinary and Universal Kriging, Kriging with an External Drift or Cokriging ([Goovaerts 1997](#), [Burrough and McDonnell 1998](#)); and
- (3) combined methods such as: Trend Surface Analysis Combined with Kriging ([Wang et al. 2005](#)), Regression Kriging ([Hengl et al. 2007](#)) and Stochastic Interpolation ([Sauquet et al. 2000](#)).

[Koutsoyiannis \(2000\)](#) presented the so-called Broken Line Smoothing (BLS) as a simple alternative to numerical smoothing and interpolating methods, related to piecewise linear regression and to smoothing splines. The idea was to approximate a smooth curve that may be drawn for the data points  $(x_i, y_i)$  with a broken line or open polygon which can be numerically estimated by means of a least squares fitting

procedure. The abscissae of the vertices of the broken line did not necessarily coincide with  $x_i$ 's but they form a series of points with some chosen, lower or higher, resolution.

Malamos and Koutsoyiannis (2014) extended the previous method by utilizing the combination of two broken lines into a regression model with known break points and adjustable weights (BLSI). The first broken line was fitted to the available data points while the second incorporates, in an objective manner, the influence of an explanatory variable available at a considerably denser dataset. The objective was to improve the accuracy of interpolation across the data points.

The concept, for both methodologies, was the trade-off between the two objectives of minimizing the fitting error and the roughness of the broken lines. The larger the relative weight of the second objective is, the smoother the broken lines resulting by the fitting procedure will be.

In the present study the method is generalized for the case of 2D data. The main idea, presented as Bilinear Surface Smoothing (BSS), is to approximate a surface that may be drawn for the data points  $(x_i, y_i)$  with consecutive bilinear surfaces which can be estimated by means of a least square fitting procedure into a surface regression model with known break points and adjustable weights. The concept was, once more, the trade-off between the two objectives of minimizing the fitting error and the roughness of the bilinear surface.

Additionally, a second version of the methodology (BSSE) is presented, which is focused on the combination of two bilinear surfaces into the same regression model, in order to improve the interpolation accuracy across the data points. The first surface is fitted to the available data points while the second incorporates, in an objective manner, the influence of an explanatory variable available at a spatially denser dataset.

The estimation of parameters, i.e. the number of surface segments and the values of the corresponding smoothing parameters, is accomplished by a nonparametric approach based on the generalized cross-validation (GCV) methodology (Craven and Wahba 1979, Wahba and Wendelberger 1980) and the linear smoothers theory (Buja *et al.* 1989). The simplified but efficient parameter estimation technique was established after numerical investigation and contributed to performance enhancement and accuracy of the mathematical framework.

## MATHEMATICAL FRAMEWORK

### Bilinear surface smoothing interpolation (BBS)

Let  $z_i(x_i, y_i)$  be a set of  $n$  points at the three dimensional space  $(x, y, z)$  for  $i = 1, \dots, n$ . Also, let  $cx_l, l = 0, \dots, mx$ , be  $mx+1$  points on the  $x$ -axis and  $cy_k, k = 0, \dots, my$ , be  $my+1$  points of the  $y$ -axis, so that the rectangle, with vertices  $(cx_0, cy_0), (cx_{mx}, cy_0), (cx_0, cy_{my})$  and  $(cx_{mx}, cy_{my})$  contain all  $(x_i, y_i)$ . For simplicity we will assume that the points on both axes are equidistant, i.e.  $cx_l - cx_{l-1} = \delta_x$  and  $cy_k - cy_{k-1} = \delta_y$ .

We wish to find the  $m + 1$  values of  $d_j$  (where  $j = 0, \dots, m$  and  $m = (mx + 1)(my + 1) - 1$ ), on the three dimensional space  $(x, y, z)$ , so that the bilinear surface defined by the  $m+1$  points  $(cx_l, cy_k, d_j)$  'fit' the set of points  $z_i(x_i, y_i)$ . This fit is defined in terms of minimizing the total square error among the set of original points  $z_i(x_i, y_i)$  and the fitted bilinear surface:

$$p = \sum_{i=1}^n (z_i - \hat{z}_i)^2 \quad (1)$$

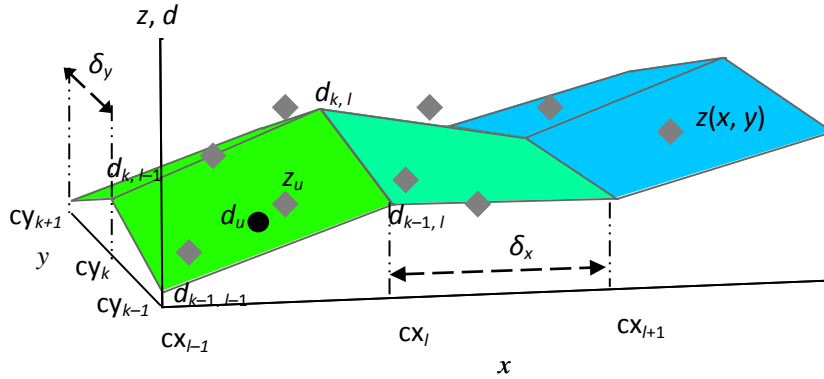
where  $\hat{z}_i$  is the estimate given by the bilinear surface for each known  $z_i$ .

In matrix form, this can be written as:

$$p = \|z - \hat{z}\|^2 \quad (2)$$

where  $z = [z_1, \dots, z_n]^T$  is the vector of known applicates of the given data points with size  $n$  (the superscript T denotes the transpose of a matrix or vector) and

$\hat{z} = [\hat{z}_1, \dots, \hat{z}_n]^T$  is the vector of estimates with size  $n$ .



**Fig. 1** Definition sketch for bilinear surface  $d$ , similar for bilinear surface  $e$

The general estimation function will be:

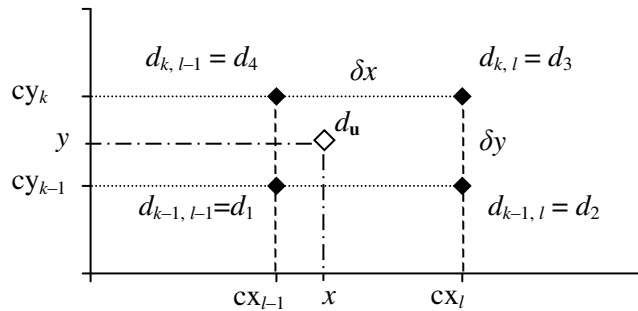
$$\hat{z}_u = d_u \quad (3)$$

where  $u$  refers to a point on the  $(x, y)$  plane, while  $d_u$  is the value of the bilinear surface at that point (Fig. 1).

The relation of  $d_u$  to its four surrounding points,  $d_1, \dots, d_4$ , as presented in Fig. 2, is simply an application of bilinear interpolation (Press *et al.* 2002):

$$d_u = \frac{1}{\delta_x \delta_y} [d_1 (cx_l - x) (cy_k - y) + d_2 (x - cx_{l-1}) (cy_k - y) + d_3 (x - cx_{l-1}) (y - cy_{k-1}) + d_4 (cx_l - x) (y - cy_{k-1})] \quad (4)$$

where  $cy_k, cy_{k-1}, cx_l, cx_{l-1}$  are the coordinates of the four points and  $x, y$ , are the corresponding coordinates of  $d_u$ . Notice that the bilinear function in (4) is not actually linear with respect to  $x$  and  $y$  as it contains products thereof.



**Fig. 2** Definition sketch for the  $d_u$  calculation

Assuming that a point  $z_i(x_i, y_i)$ , lies in the two-dimensional space ( $[cx_{l-1}, cx_l] \times [cy_{k-1}, cy_k]$ ) for some  $cx_l$ , ( $cx_{l-1} \leq x_i \leq cx_l$ ) and some  $cy_k$ , ( $cy_{k-1} \leq y_i \leq cy_k$ ), then obviously the  $\hat{z}_i$  estimate is given by:

$$\begin{aligned} \hat{z}_i(x_i, y_i) = & \frac{1}{\delta_x \delta_y} [d_1 (cx_l - x_i) (cy_k - y_i) + d_2 (x_i - cx_{l-1}) (cy_k - y_i) + \\ & + d_3 (x_i - cx_{l-1}) (y_i - cy_{k-1}) + d_4 (cx_l - x_i) (y_i - cy_{k-1})] \end{aligned} \quad (5)$$

If we apply equation (5) for  $i = 1, \dots, n$ , we get:

$$\begin{aligned} \hat{z}_1 = & \frac{1}{\delta_x \delta_y} [d_1 (cx_1 - x_1) (cy_1 - y_1) + d_2 (x_1 - cx_0) (cy_1 - y_1) + \\ & + d_3 (x_1 - cx_0) (y_1 - cy_0) + d_4 (cx_1 - x_1) (y_1 - cy_0)] \\ & \vdots \\ \hat{z}_n = & \frac{1}{\delta_x \delta_y} [d_{k-1, l-1} (cx_l - x_n) (cy_k - y_n) + d_{k, l-1} (x_n - cx_{l-1}) (cy_k - y_n) + \\ & + d_{k, l} (x_n - cx_{l-1}) (y_n - cy_{k-1}) + d_{k-1, l} (cx_l - x_n) (y_n - cy_{k-1})] \end{aligned} \quad (6)$$

The above equations can be more concisely written in the form:

$$\hat{\mathbf{z}} = \mathbf{H} \mathbf{d} \quad (7)$$

where  $\hat{\mathbf{z}} = [\hat{z}_1, \dots, \hat{z}_n]^T$  is the vector of estimates with size  $n$ ;  $\mathbf{d} = [d_0, \dots, d_m]^T$  is the vector of the unknown applicates of the bilinear surface  $d$  with size  $m+1$  ( $m = (mx + 1)(my + 1) - 1$ ) and  $\mathbf{H}$  is a matrix with size  $n \times (m+1)$  whose  $ij$ th entry (for  $i=1, \dots, n; j=0, \dots, m$ ) is:

$$\pi_{ij} = \begin{cases} \frac{(cx_l - x_i) (cy_k - y_i)}{\delta_x \delta_y}, & \text{when } cx_{l-1} < x_i \leq cx_l \text{ and } cy_{k-1} < y_i \leq cy_k \\ \frac{(cx_l - x_i) (y_i - cy_{k-1})}{\delta_x \delta_y}, & \text{when } cx_{l-1} < x_i \leq cx_l \text{ and } cy_k \leq y_i < cy_{k+1} \\ \frac{(x_i - cx_{l-1}) (y_i - cy_{k-1})}{\delta_x \delta_y}, & \text{when } cx_l \leq x_i < cx_{l+1} \text{ and } cy_k \leq y_i < cy_{k+1} \\ \frac{(x_i - cx_{l-1}) (cy_k - y_i)}{\delta_x \delta_y}, & \text{when } cx_l \leq x_i < cx_{l+1} \text{ and } cy_{k-1} < y_i \leq cy_k \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

In order to acquire the amount of smoothness of the bilinear surface  $d$  and to assure a unique solution of the fitting problem, we introduced the difference of slopes between two consecutive segments of the bilinear surface according to  $x$  direction, for each  $cy_k$  point on the  $y$ -axis, by taking into account the fact that  $cx_l$ 's are equidistant, as:

$$\frac{1}{\delta_x} (2d_{l, k} - d_{l-1, k} - d_{l+1, k}) \quad (9)$$

Likewise for the  $y$  direction, for each  $cx_l$  point on the  $x$ -axis, by taking into account the fact that  $cy_k$ 's are equidistant, the slope difference will be:

$$\frac{1}{\delta_y} (2d_{k,l} - d_{k-1,l} - d_{k+1,l}) \quad (10)$$

Therefore, the following expressions constitute adequate smoothing terms of the bilinear surface for both directions:

$$q_{dx} = \sum_{k=0}^{my} \sum_{l=1}^{mx-1} (2d_{l,k} - d_{l-1,k} - d_{l+1,k})^2 \quad (11)$$

and

$$q_{dy} = \sum_{l=0}^{mx} \sum_{k=1}^{my-1} (2d_{k,l} - d_{k-1,l} - d_{k+1,l})^2 \quad (12)$$

which can easily be expressed in matrix form as follows:

$$q_{dx} = \mathbf{d}^T \boldsymbol{\Psi}_x^T \boldsymbol{\Psi}_x \mathbf{d} \quad (13)$$

$$q_{dy} = \mathbf{d}^T \boldsymbol{\Psi}_y^T \boldsymbol{\Psi}_y \mathbf{d} \quad (14)$$

where  $\boldsymbol{\Psi}_x$  and  $\boldsymbol{\Psi}_y$  are matrices with size  $(m-1) \times (m+1)$  (for  $i=1, \dots, m-1$  and  $j=0, \dots, m$ ). As explained in Appendix A, their  $ij$ th entry is:

$$\psi_{x,i,j} = \begin{cases} 2, & \text{when } i=j \text{ and } i-k(mx+1) \notin \{1, mx+1\} \\ -1, & \text{when } |i-j|=1 \text{ and } i-k(mx+1) \notin \{1, mx+1\} \\ 0, & \text{otherwise} \end{cases} \quad (15)$$

where  $k = 0, \dots, my$ , while

$$\psi_{y,i,j} = \begin{cases} 2, & \text{when } i=j \text{ and } i-l(my+1) \notin \{1, my+1\} \\ -1, & \text{when } |i-j|=1 \text{ and } i-l(my+1) \notin \{1, my+1\} \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

with  $l = 0, \dots, mx$ . It is noted that matrices  $\boldsymbol{\Psi}_x$  and  $\boldsymbol{\Psi}_y$  are identical when  $mx = my$ .

Combining equations (2), (7), (13), (14) and introducing dimensionless multipliers for both  $x$  and  $y$  directions in order to control the smoothness of the bilinear surface, we form the generalized objective function to be minimized:

$$f(\mathbf{d}) = p + \lambda_x q_{dx} + \lambda_y q_{dy} = \|\mathbf{z} - \hat{\mathbf{z}}\|^2 + \lambda_x \mathbf{d}^T \boldsymbol{\Psi}_x^T \boldsymbol{\Psi}_x \mathbf{d} + \lambda_y \mathbf{d}^T \boldsymbol{\Psi}_y^T \boldsymbol{\Psi}_y \mathbf{d} \quad (17)$$

where  $\lambda_x \geq 0$  for  $q_{dx}$  and  $\lambda_y \geq 0$  for  $q_{dy}$ .

Differentiation of equation (17) with respect to  $\mathbf{d}$ , by applying the typical rules of derivatives involving matrices and equating to zero yields:

$$\frac{\partial f}{\partial \mathbf{d}} = -2\mathbf{z}^T \boldsymbol{\Pi} + 2\mathbf{d}^T \boldsymbol{\Pi}^T \boldsymbol{\Pi} + 2\lambda_x \mathbf{d}^T \boldsymbol{\Psi}_x^T \boldsymbol{\Psi}_x + 2\lambda_y \mathbf{d}^T \boldsymbol{\Psi}_y^T \boldsymbol{\Psi}_y = 0 \quad (18)$$

and consequently:

$$(\boldsymbol{\Pi}^T \boldsymbol{\Pi} + \lambda_x \boldsymbol{\Psi}_x^T \boldsymbol{\Psi}_x + \lambda_y \boldsymbol{\Psi}_y^T \boldsymbol{\Psi}_y) \mathbf{d} = \boldsymbol{\Pi}^T \mathbf{z} \quad (19)$$

Finally, the solution of equation (19) that minimizes equation (17), has the following form:

$$\mathbf{d} = (\boldsymbol{\Pi}^T \boldsymbol{\Pi} + \lambda_x \boldsymbol{\Psi}_x^T \boldsymbol{\Psi}_x + \lambda_y \boldsymbol{\Psi}_y^T \boldsymbol{\Psi}_y)^{-1} \boldsymbol{\Pi}^T \mathbf{z} \quad (20)$$

The vector of estimates,  $\hat{z}$ , is obtained from equation (7), once vector  $d$  is calculated from equation (20). Also, from equation (5), we can estimate the applicator  $\hat{z}$  of any point that lies in the two-dimensional interval  $([cx_0, cx_{mx}] \times [cy_0, cy_{my}])$ .

The minimum number of  $m + 1$  points required to solve equation (20) is 6, according to equations (11) and (12). This is illustrated in Fig. 1 since the minimum number of points needed to define the bilinear surface  $d$ , is the number of points that define two consecutive planes oriented according to either  $x$  or  $y$  direction.

### **Bilinear surface smoothing interpolation with the incorporation of explanatory variable (BSSE)**

The incorporation of an explanatory variable available at a considerably denser dataset than the initial main variable constitutes a distinct interpolation method that extends the above presented mathematical framework. The methodology is based on the one-dimensional implementation presented by Malamos and Koutsoyiannis (2014).

Let  $z_i(x_i, y_i)$  be the same set of  $n$  points at the three dimensional space  $(x, y, z)$  for  $i = 1, \dots, n$ , as already defined in the previously presented case.

In addition, we assume that for every  $(x, y)$  value we know the value of an explanatory variable  $t$ . Therefore, for each point  $z_i(x_i, y_i)$  there corresponds a value  $t(x_i, y_i)$  and for point  $(cx_l, cy_k)$  there corresponds a value  $t(cx_l, cy_k)$ , for  $l = 0, \dots, mx$ , and  $k = 0, \dots, my$ .

We wish to find the  $m + 1$  values of  $d_j$  and  $e_j$  (where  $j = 0, \dots, m$  and  $m = (mx + 1)(my + 1) - 1$ ), on the three dimensional space  $(x, y, z)$ , so that the bilinear surface defined by the  $m+1$  points  $[cx_l, cy_k, d_j + t(cx_l, cy_k) \times e_j]$  fits the set of points  $z_i(x_i, y_i)$ . This fit is meant in terms of minimizing the total square error among the set of original points  $z_i(x_i, y_i)$  and the fitted bilinear surface as already presented in equations (1) and (2).

In this case, the general estimation function will be:

$$\hat{z}_u = d_u + t_u e_u \quad (21)$$

where  $u$  refers to a point on the  $(x, y)$  plane, while  $d_u, e_u$  are the values of the two bilinear surfaces at that point and  $t_u$  is the corresponding value of the explanatory variable. This is not a global linear relationship but a local linear one as the quantities  $d_u$  and  $e_u$  change with  $x$  and  $y$ .

Following the methodology presented above, we obtain the relation that provides the second bilinear surface,  $e_u$ , which is:

$$e_u = \frac{1}{\delta_x \delta_y} [e_1 (cx_l - x) (cy_k - y) + e_2 (x - cx_{l-1}) (cy_k - y) + e_3 (x - cx_{l-1}) (y - cy_{k-1}) + e_4 (cx_l - x) (y - cy_{k-1})] \quad (22)$$

Assuming that a point  $z_i(x_i, y_i)$ , lies in the two-dimensional interval  $[cx_{l-1}, cx_l] \times [cy_{k-1}, cy_k]$  for some  $cx_l$ ,  $(cx_{l-1} \leq x_i \leq cx_l)$  and some  $cy_k$ ,  $(cy_{k-1} \leq y_i \leq cy_k)$ , then the  $\hat{z}_i$  estimate is given by:

$$\begin{aligned} \hat{z}_i(x_i, y_i) = & \frac{1}{\delta_x \delta_y} \{ [d_1 (cx_l - x_i) (cy_k - y_i) + d_2 (x_i - cx_{l-1}) (cy_k - y_i) + \\ & + d_3 (x_i - cx_{l-1}) (y_i - cy_{k-1}) + d_4 (cx_l - x_i) (y_i - cy_{k-1})] \\ & + t(x_i, y_i) [e_1 (cx_l - x_i) (cy_k - y_i) + e_2 (x_i - cx_{l-1}) (cy_k - y_i) + \end{aligned}$$

$$+ e_3 (x_i - cx_{l-1}) (y_i - cy_{k-1}) + e_4 (cx_l - x_i) (y_i - cy_{k-1}) \} \quad (23)$$

If we apply equation (23) for  $i = 1, \dots, n$ , we obtain the following form, analogous to equation (6):

$$\begin{aligned} \hat{z}_1 &= \frac{1}{\delta_x \delta_y} \{ [d_1 (cx_1 - x_1) (cy_1 - y_1) + d_2 (x_1 - cx_0) (cy_1 - y_1) + \\ &+ d_3 (x_1 - cx_0) (y_1 - cy_0) + d_4 (cx_1 - x_1) (y_1 - cy_0)] + \\ &+ t(x_1 y_1) [e_1 (cx_1 - x_1) (cy_1 - y_1) + e_2 (x_1 - cx_0) (cy_1 - y_1) + \\ &+ e_3 (x_1 - cx_0) (y_1 - cy_0) + e_4 (cx_1 - x_1) (y_1 - cy_0)] \} \\ &\quad \vdots \\ \hat{z}_n &= \frac{1}{\delta_x \delta_y} \{ [d_{k-1, l-1} (cx_l - x_n) (cy_k - y_n) + d_{k, l-1} (x_n - cx_{l-1}) (cy_k - y_n) + \\ &+ d_{k, l} (x_n - cx_{l-1}) (y_n - cy_{k-1}) + d_{k-1, l} (cx_l - x_n) (y_n - cy_{k-1})] + \\ &+ t(x_n y_n) [e_{k-1, l-1} (cx_1 - x_n) (cy_1 - y_n) + e_{k, l-1} (x_n - cx_0) (cy_1 - y_n) + \\ &+ e_{k, l} (x_n - cx_{l-1}) (y_n - cy_{k-1}) + e_{k-1, l} (cx_l - x_n) (y_n - cy_{k-1})] \} \end{aligned} \quad (24)$$

This can be more concisely written in matrix form as:

$$\hat{\mathbf{z}} = \mathbf{\Pi} \mathbf{d} + \mathbf{T} \mathbf{\Pi} \mathbf{e} \quad (25)$$

where  $\hat{\mathbf{z}} = [\hat{z}_1, \dots, \hat{z}_n]^T$  is the vector of estimates with size  $n$ ;  $\mathbf{d} = [d_0, \dots, d_m]^T$  is the vector of the unknown applicates of the bilinear surface  $d$ , with size  $m+1$  ( $m = (mx + 1) \times (my + 1) - 1$ );  $\mathbf{e} = [e_0, \dots, e_m]^T$  is the vector of the unknown applicates of the bilinear surface  $e$ , with size  $m+1$ ;  $\mathbf{T}$  is a  $n \times n$  diagonal matrix:

$$\mathbf{T} = \text{diag}(t(x_1, y_1), \dots, t(x_n, y_n)) \quad (26)$$

with its elements  $t(x_1, y_1), \dots, t(x_n, y_n)$  being the values of the explanatory variable at the given data points; and  $\mathbf{\Pi}$  is a matrix with size  $n \times (m+1)$  as defined in equation (8).

In order to incorporate the amount of smoothness of the second bilinear surface  $e$  and following the procedure presented in equations (9) to (12), we conclude to the following expressions for the smoothness of the bilinear surface  $e$  at  $x$  and  $y$  directions:

$$q_{\text{ex}} = \sum_{k=0}^{my} \sum_{l=1}^{mx-1} (2e_{l, k} - e_{l-1, k} - e_{l+1, k})^2 \quad (27)$$

and

$$q_{\text{ey}} = \sum_{l=0}^{mx} \sum_{k=1}^{my-1} (2e_{k, l} - e_{k-1, l} - e_{k+1, l})^2 \quad (28)$$

In matrix form, equations (27) and (28) along with equations (11) and (12), express the amount of smoothness of the bilinear surfaces  $d, e$ , for the BSSE case, as follows:

$$q_{\text{dx}} = \mathbf{d}^T \mathbf{\Psi}_x^T \mathbf{\Psi}_x \mathbf{d}, \quad q_{\text{dy}} = \mathbf{d}^T \mathbf{\Psi}_y^T \mathbf{\Psi}_y \mathbf{d} \quad (29)$$

$$q_{\text{ex}} = \mathbf{e}^T \mathbf{\Psi}_x^T \mathbf{\Psi}_x \mathbf{e}, \quad q_{\text{ey}} = \mathbf{e}^T \mathbf{\Psi}_y^T \mathbf{\Psi}_y \mathbf{e} \quad (30)$$

where  $\Psi_x$  and  $\Psi_y$  are matrices with size  $(m-1) \times (m+1)$  (for  $i=1, \dots, m-1$  and  $j=0, \dots, m$ ) and  $ij$ th entry as in equations (15) and (16) respectively (see Appendix A).

Combining equations (2), (25), (29), (30) and introducing dimensionless multipliers for both  $x$  and  $y$  directions in order to control the smoothness of the bilinear surfaces, we form the generalized objective function to be minimized:

$$\begin{aligned} f(\mathbf{d}, \mathbf{e}) := & p + \lambda_x q_{dx} + \lambda_y q_{dy} + \mu_x q_{ex} + \mu_y q_{ey} = \\ & \|\mathbf{z} - \hat{\mathbf{z}}\|^2 + \lambda_x \mathbf{d}^T \Psi_x^T \Psi_x \mathbf{d} + \lambda_y \mathbf{d}^T \Psi_y^T \Psi_y \mathbf{d} + \\ & + \mu_x \mathbf{e}^T \Psi_x^T \Psi_x \mathbf{e} + \mu_y \mathbf{e}^T \Psi_y^T \Psi_y \mathbf{e} \end{aligned} \quad (31)$$

where  $\lambda_x \geq 0$  for  $q_{dx}$ ,  $\lambda_y \geq 0$  for  $q_{dy}$  and  $\mu_x \geq 0$  for  $q_{ex}$ ,  $\mu_y \geq 0$  for  $q_{ey}$ .

Differentiation of equation (31) with respect to  $\mathbf{d}$  and  $\mathbf{e}$ , by applying the typical rules of derivatives involving matrices and equating them to zero yields:

$$\frac{\partial f_1}{\partial \mathbf{d}} = -2\mathbf{z}^T \Pi + 2\mathbf{d}^T \Pi^T \Pi + 2\mathbf{e}^T \Pi^T T^T \Pi + 2\lambda_x \mathbf{d}^T \Psi_x^T \Psi_x + 2\lambda_y \mathbf{d}^T \Psi_y^T \Psi_y = 0 \quad (32)$$

$$\frac{\partial f_2}{\partial \mathbf{e}} = -2\mathbf{z}^T T \Pi + 2\mathbf{d}^T \Pi^T T \Pi + 2\mathbf{e}^T \Pi^T T^T T \Pi + 2\mu_x \mathbf{e}^T \Psi_x^T \Psi_x + 2\mu_y \mathbf{e}^T \Psi_y^T \Psi_y = 0 \quad (33)$$

and consequently:

$$\begin{aligned} & [\Pi^T \Pi + \lambda_x \Psi_x^T \Psi_x + \lambda_y \Psi_y^T \Psi_y] \mathbf{d} + \Pi^T T \Pi \mathbf{e} = \Pi^T \mathbf{z} \\ & \Pi^T T \Pi \mathbf{d} + [\Pi^T T^T T \Pi + \mu_x \Psi_x^T \Psi_x + \mu_y \Psi_y^T \Psi_y] \mathbf{e} = \Pi^T T^T \mathbf{z} \end{aligned} \quad (34)$$

Finally, the solution of the above set of equations that provides the unknown vectors  $\mathbf{d}$ ,  $\mathbf{e}$  which minimize equation (31), is:

$$\begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \Pi^T \Pi + \lambda_x \Psi_x^T \Psi_x + \lambda_y \Psi_y^T \Psi_y & \Pi^T T \Pi \\ \Pi^T T \Pi & \Pi^T T^T T \Pi + \mu_x \Psi_x^T \Psi_x + \mu_y \Psi_y^T \Psi_y \end{bmatrix}^{-1} \begin{bmatrix} \Pi^T \mathbf{z} \\ \Pi^T T^T \mathbf{z} \end{bmatrix} \quad (35)$$

The vector of estimates,  $\hat{\mathbf{z}}$ , is obtained from equation (25), once vectors  $\mathbf{d}$  and  $\mathbf{e}$  are calculated from equation (35). Also, from equation (23), we can estimate the applicate  $\hat{z}$  of any point that lies in the two-dimensional interval  $[cx_0, cx_{mx}] \times [cy_0, cy_{my}]$ .

We observe that from the four matrices with size  $(m+1) \times (m+1)$  appearing in equations (20) and (35), i.e.  $\mathbf{B} := \Pi^T \Pi$ ,  $\mathbf{C} := \Psi_x^T \Psi_x$ ,  $\mathbf{D} := \Pi^T T^T T \Pi$  and  $\mathbf{E} := \Psi_y^T \Psi_y$ ;  $\mathbf{B}$  and  $\mathbf{D}$  are symmetric block tridiagonal while  $\mathbf{C}$  and  $\mathbf{E}$  are block diagonal matrices. Furthermore, matrices  $\mathbf{C}$  and  $\mathbf{E}$  are always singular, however when  $\lambda_x, \mu_x > 0$  or  $\lambda_y, \mu_y > 0$ , the sums  $\mathbf{B} + \lambda_x \mathbf{C} + \lambda_y \mathbf{E}$  and  $\mathbf{D} + \mu_x \mathbf{C} + \mu_y \mathbf{E}$  are non-singular and thus, their inverses exist.

## CHOICE OF PARAMETERS

### Transformation of smoothing parameters

It is apparent that the number of the adjustable parameters for each of the two above presented versions of the methodology consists of the numbers of intervals,  $mx$ ,  $my$ , and the smoothing parameters for the  $x$ ,  $y$  directions.

Therefore for the case of the bilinear surface interpolation (BSS) there are four adjustable parameters: the numbers of intervals,  $mx$ ,  $my$ , and the smoothing parameters  $\lambda_x$  and  $\lambda_y$  corresponding to vector  $\mathbf{d}$ . The incorporation of the explanatory



variable, for the BSSE case, adds two more adjustable parameters: the smoothing parameters  $\mu_x$  and  $\mu_y$  corresponding to vector  $e$ .

The choice of parameters can be done by using an efficient, but standard objective way as described by the following analysis:

A convenient search of the smoothing parameters, in terms of computational time, can be achieved by transforming  $\lambda$  and  $\mu$  in terms of tension parameters  $\tau_\lambda$  and  $\tau_\mu$ , whose values are restricted in the interval  $[0, 1)$ , for both directions. The formulation is based on the expressions presented by Koutsoyiannis (2000), as well as Malamos and Koutsoyiannis (2014), and was established after a numerical investigation of the method on several examples. The proposed equations have the form:

$$\lambda_x = \left(10^\varepsilon m \frac{\log \tau_m}{\log \tau_{\lambda x}}\right) \kappa_\lambda, \lambda_y = \left(10^\varepsilon m \frac{\log \tau_m}{\log \tau_{\lambda y}}\right) \kappa_\lambda \quad (36)$$

for the BSS case, while for BSSE the extra smoothing parameters  $\mu_x$  and  $\mu_y$  are set to:

$$\mu_x = \left(10^\theta m \frac{\log \tau_m}{\log \tau_{\mu x}}\right) \kappa_\mu, \mu_y = \left(10^\theta m \frac{\log \tau_m}{\log \tau_{\mu y}}\right) \kappa_\mu \quad (37)$$

where  $\tau_m = 0.99$  is the maximum allowed tension, corresponding to the upper bound of  $\lambda$  and  $\mu$ , set for numerical stability equal to:

$$\lambda_m = \frac{\text{trace}(\mathbf{B})}{\text{trace}(\mathbf{C} + \mathbf{E})} 10^9, \mu_m = \frac{\text{trace}(\mathbf{D})}{\text{trace}(\mathbf{C} + \mathbf{E})} 10^9 \quad (38)$$

The exponents  $\kappa_\lambda, \kappa_\mu$  in equations (36), (37) are determined by the relations:

$$\kappa_\lambda = \frac{\log \lambda_m}{\log(10^\varepsilon m)}, \kappa_\mu = \frac{\log \mu_m}{\log(10^\theta m)}, m = (m_x + 1)(m_y + 1) - 1 \quad (39)$$

which are obtained by combining equations (36) or (37) with equation (38). The exponents  $\varepsilon, \theta$  in equations (36), (37) and (39) are set to:

$$\varepsilon = \max(1, \lfloor \log [\text{trace}(\mathbf{B})] \rfloor) \quad (40)$$

and

$$\theta = \max(1, \lfloor \log [\text{trace}(\mathbf{D})] \rfloor) \quad (41)$$

with  $\varepsilon, \theta \in \mathbb{Z}^+$ . The minimum allowed values of  $\lambda_x, \lambda_y, \mu_x, \mu_y$  is 0.

### Estimation of smoothing parameters

Combining equations (7) and (20) for the BSS case, we obtain:

$$\hat{\mathbf{z}} = \mathbf{A} \mathbf{z} \quad (42)$$

where  $\mathbf{A}$  is a  $n \times n$  symmetric matrix given by:

$$\mathbf{A} = \mathbf{\Pi} (\mathbf{\Pi}^T \mathbf{\Pi} + \lambda_x \mathbf{\Psi}_x^T \mathbf{\Psi}_x + \lambda_y \mathbf{\Psi}_y^T \mathbf{\Psi}_y)^{-1} \mathbf{\Pi}^T \quad (43)$$

while combining equations (25) and (35) for the case with explanatory variable (BSSE) we obtain the same relationship as equation (42) with  $\mathbf{A}$  being a  $n \times n$  symmetric matrix now given by:

$$\mathbf{A} = \mathbf{\Pi} \mathbf{T} \mathbf{\Pi}$$

$$\left[ \begin{array}{cc} \mathbf{\Pi}^T \mathbf{\Pi} + \lambda_x \mathbf{\Psi}_x^T \mathbf{\Psi}_x + \lambda_y \mathbf{\Psi}_y^T \mathbf{\Psi}_y & \mathbf{\Pi}^T \mathbf{T} \mathbf{\Pi} \\ \mathbf{\Pi}^T \mathbf{T} \mathbf{\Pi} & \mathbf{\Pi}^T \mathbf{T}^T \mathbf{T} \mathbf{\Pi} + \mu_x \mathbf{\Psi}_x^T \mathbf{\Psi}_x + \mu_y \mathbf{\Psi}_y^T \mathbf{\Psi}_y \end{array} \right]^{-1}$$

$$(\mathbf{II} \mathbf{III})^T \quad (44)$$

Equations (43) and (44) depend on all adjustable parameters:  $m_x$ ,  $m_y$ ,  $\tau_{\lambda_x}$ ,  $\tau_{\lambda_y}$  and  $\tau_{\mu_x}$ ,  $\tau_{\mu_y}$ .

The parameter estimation is based on the generalized cross-validation ([Craven and Wahba 1979](#)) methodology, defined by:

$$\text{GCV} = \frac{\frac{1}{n} \|(I-A)z\|^2}{\left[ \frac{1}{n} \text{trace}(I-A) \right]^2} \quad (45)$$

where matrix  $A$  is called “influence” or “smoother” matrix, while the quantity:

$$\text{trace}(I-A) \quad (46)$$

in the denominator of equation (45) describes the “residual degrees of freedom” of the fitted smoother used by nonparametric regression methods ([Buja et al. 1989](#), [Wahba 1990](#), [Carmack et al. 2012](#)).

Based on literature, there are two alternative definitions for residual degrees of freedom under independence in the context of symmetric linear smoothers, namely:

$$\text{trace}(I-AA^T) \quad (47)$$

$$\text{trace}[I-(2A-AA^T)] \quad (48)$$

with  $0 \leq \text{trace}[I-(2A-AA^T)] \leq \text{trace}(I-A) \leq \text{trace}(I-AA^T) \leq n$  ([Buja et al. 1989](#), [Carmack et al. 2012](#)).

For exploration purposes, we analyzed the methods’ performance against all three definitions. The results showed that when matrix  $A$  is defined by equation (43), the best results were obtained when the residual degrees of freedom were defined by equation (47). However, when matrix  $A$  is defined by equation (44), the best results were obtained when the residual degrees of freedom were defined by equation (46), which is the standard definition of the generalized cross-validation, as already presented in equation (45). The degrees of freedom definition presented by equation (48) did not perform as well as the previous mentioned expressions and thus it was excluded from the methods’ implementation.

Consequently, the relation that provides GCV for the BSS method is:

$$\text{GCV} = \frac{\frac{1}{n} \|(I-A)z\|^2}{\left[ \frac{1}{n} \text{trace}(I-AA^T) \right]^2} \quad (49)$$

while equation (45) is used for estimating GCV for the BSSE method.

Based on the above presented analysis, for a given combination of segments  $m_x$ ,  $m_y$  the minimization of GCV, results in the optimum values of  $\tau_{\lambda_x}$ ,  $\tau_{\lambda_y}$  and  $\tau_{\mu_x}$ ,  $\tau_{\mu_y}$ . This can be repeated for several trial combinations of  $m_x$ ,  $m_y$  values, until the global minimum of GCV is reached.

## COMPUTATIONAL IMPLEMENTATION

In similar applications presented earlier by [Koutsoyiannis \(2000\)](#) and [Malamos and Koutsoyiannis \(2014\)](#), the implementation of the computational framework was made in Microsoft Excel since it provides a direct means of data visualization and graphical exploration.

Since the block matrices involved in the systems of equations (20) and (35) have dimensions  $(m + 1) \times (m + 1)$  and  $(2m + 2) \times (2m + 2)$ , respectively, a considerable computational effort that could not be satisfied from Microsoft Excel alone, was required. This was tackled by the development of a dynamic link library in Object Pascal (Delphi) programming language, which was linked to Microsoft Excel.

In this context, an Excel array formula acts as the main interface, with its arguments being the available points' values and coordinates along with the unknown points' coordinates, the number of points on the  $x$  and  $y$  axis that form the bilinear surfaces and the smoothing parameters values.

The dynamic link library performs the following tasks:

- (1) constructs the matrices involved in the systems of equations (20) or (35), depending on which of the two versions of the methodology is implemented.
- (2) solves the system of equations, for each case, by decomposition into upper and lower triangular matrices, known as "LU Decomposition" method.
- (3) finds the inverse matrices involved in equations (43) and (44) by a straightforward procedure based on the above mentioned "LU Decomposition" method (Press *et al.* 2002).
- (4) returns to Microsoft Excel apart from the solution of the systems of equations (20) or (35), information concerning the above presented numerical procedure, such as: the matrices  $\mathbf{B}$ ,  $\mathbf{C}$ ,  $\mathbf{D}$  and  $\mathbf{E}$ , along with the GCV and mean square estimation error. The latter is acquired from the numerators of equations (45) and (49).

## RESULTS AND COMMENTS

The BSS and BSSE methods, with the mathematical formulation described in the previous sections, were derived from extending in two dimensions, the Broken Line Smoothing method described by Koutsoyiannis (2000) and the Broken Line Smoothing with explanatory variable described by Malamos and Koutsoyiannis (2014).

The main difference between bilinear surface smoothing methods and other known interpolation methods is the introduction of the smoothness terms  $\Psi_x^T \Psi_x$  and  $\Psi_y^T \Psi_y$  in the corresponding problem formulation. Those terms control the overall smoothness of the bilinear surface through adjustable parameters according to  $x$  or  $y$  direction.

It should be obvious from the above discourse that bilinear surface smoothing methods do not require linearity between the involved variables, namely  $x$ ,  $y$ ,  $z$  and the explanatory variable  $t$ , but two-dimensional local bilinearity is incorporated in the mathematical framework in a bilinear surface approach. Also, the functional dependence, in terms of vectors  $\mathbf{d}$ ,  $\mathbf{e}$ , the number of segments,  $m_x$  and  $m_y$ , and the tension parameters, is neither constant nor a priori known, but in each case is determined through the procedure of minimizing the generalized cross-validation (GCV).

Both implementations of bilinear surface smoothing require the minimization of generalized objective functions with respect to the total square error and the surface smoothness. The formulation of equation (42) allows the adaptation of the generalized cross-validation from the splines theory, allowing a standard and objective way to estimate the smoothness parameters and the number of the bilinear surfaces involved in the interpolation procedure.

According to the classification presented by [Li and Heap \(2008\)](#), BSS and BSSE have the following features:

- (1) They are both *local* and *global*. Their locality stems from the fact that they use the four surrounding points of the corresponding bilinear surface to derive the estimation of the included data point (Fig. 2). On the other hand, they are also global since they implement the GCV procedure to globally fit the consecutive bilinear surfaces to the available data points.
- (2) They can be either *exact* or *inexact*. Specifically, they are able to generate an estimate that is the same as the observed value at a sampled point (exactness) if the minimum values of the smoothing parameters are used. On the other hand, when the GCV procedure is implemented along with strong smoothing, they are inexact.
- (3) They are *stochastic* since the proposed mathematical framework, apart from estimations provides also direct means of evaluating interpolation errors across the available data points from the numerators of equations (45) and (49) as already presented in the one-dimensional implementation (Malamos and Koutsoyiannis 2014).
- (4) The surfaces that they produce can be either *gradual* or *abrupt* depending on the magnitude of the smoothing parameters, e.g. if their values are close to 1, the resulting surface will be smooth while the opposite will occur if their values are close to the lower limit. Also, the numbers of bilinear surfaces along  $x$  and/or  $y$  directions, i.e.  $m_x$  and  $m_y$ , contribute to the overall surface smoothness thus acting as additional smoothing parameters. This derives also from the one-dimensional implementations (Koutsoyiannis 2000; Malamos and Koutsoyiannis 2014), where increased numbers of broken lines segments were associated with small values of the smoothing parameters.
- (5) BSS is *univariate* since it implements only the primary variable in deriving the estimation, while BSSE is *multivariate* since it incorporates an explanatory variable available at a considerably denser dataset in the interpolation procedure.
- (6) Both BSS and BSSE implement a regular grid but this does not have to be necessary square since the number of bilinear surfaces along the  $x$  direction does not have to coincide with the number of bilinear surfaces along the  $y$  direction.

## CONCLUSIONS

A non-parametric innovative mathematical framework which can be utilized to perform various interpolation tasks is described. The technique incorporates smoothing terms with adjustable weights, defined by means of the angles formed by the consecutive bilinear surfaces into a piecewise surface regression model with known break points. The incorporation, in an objective manner, of an explanatory variable available from measurements at a considerably denser dataset than the initial main variable, is presented in terms of an alternative implementation of the main methodology.

A notable property of the proposed framework is the fact that the resolution (number of consecutive bilinear surfaces) does not necessarily has to coincide with that of the given data points, but it can be either finer or coarser, depending on the specific requirements of the problem of interest. This is an important property that makes the method applicable and reliable even in the case of scarce datasets.

The proposed mathematical framework follows a parsimonious approach for fulfilling spatial interpolation tasks, without the need to make many decisions on parameters or complex concepts. Likewise, the computational implementation offers an almost automated procedure in achieving the final results.

Further research can be focused towards the incorporation of alternative techniques for acquiring the global minimum value of GCV, providing means for faster convergence to the optimal solution.

The method application in hydrological problems is given in a companion paper (Malamos and Koutsoyiannis submitted) along with comparisons to established interpolation methods.

## ACKNOWLEDGMENT

We thank the Associate Editor Alin Castreanu, the eponymous reviewer Efraín Domínguez and an anonymous reviewer for their comments and suggestions which helped us to improve our paper.

## REFERENCES

- Buja, A., Hastie, T., and Tibshirani, R., 1989. Linear Smoothers and Additive Models. *The Annals of Statistics*, 17 (2), 453–510.
- Burrough, P.A. and McDonnell, R.A., 1998. *Principles of Geographical Information Systems*. Oxford University Press, Oxford, 333 pp.
- Carmack, P.S., Spence, J.S., and Schucany, W.R., 2012. Generalised correlated cross-validation. *Journal of Nonparametric Statistics*, 24 (2), 269–282.
- Craven, P. and Wahba, G., 1979. Smoothing noisy data with spline functions. *Numerische Mathematik*, 31 (4), 377–403.
- Davis C., J., 1986. *Statistics and Data Analysis in Geology*, 2<sup>nd</sup> edition. John Wiley & Sons Canada, Ltd.
- Goovaerts, P., 1997. *Geostatistics for Natural Resources Evaluation*. Oxford University Press, New York, 483 pp.
- Hengl, T., Heuvelink, G.B.M., Rossiter, D.G., 2007. About regression-kriging: From equations to case studies. *Computers & Geosciences* 33 (10), 1301-1315.
- Koutsoyiannis, D., 2000. Broken line smoothing: a simple method for interpolating and smoothing data series. *Environmental Modelling & Software* 15 (2), 139-149.
- Li, J. and Heap, A.D., 2008. *A Review of Spatial Interpolation Methods for Environmental Scientists*. Geoscience Australia, Record 2008/23, 137 pp.
- Malamos, N. and Koutsoyiannis, D., 2014. Broken line smoothing for data series interpolation by incorporating an explanatory variable with denser observations: Application to soil-water and rainfall data. *Hydrological Sciences Journal* doi:10.1080/02626667.2014.899703.
- Malamos, N. and Koutsoyiannis, D., (submitted). Bilinear surface smoothing for spatial interpolation with optional incorporation of an explanatory variable. Part 2: Application to synthesized and rainfall data, *Hydrological Sciences Journal*.
- Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P., 2002. *Numerical recipes in C*. 2<sup>nd</sup> edition. Cambridge, Cambridge University Press.
- Sauquet, E., Gottschalk, L., and Leblois, E., 2000. Mapping average annual runoff: a hierarchical approach applying a stochastic interpolation scheme. *Hydrological Sciences Journal*, 45 (6), 799–815.
- Wahba, G., Wendelberger, J., 1980. Some New Mathematical Methods for Variational Objective Analysis Using Splines and Cross Validation. *Monthly Weather Review* 108 (8), 1122-1143.

- Wahba, G., 1990. *Spline Models for Observational Data*. Regional Conference Series in Applied Mathematics 59. Philadelphia, Pennsylvania, SIAM.
- Wang, H., Liu, G. and Gong, P., 2005. Use of cokriging to improve estimates of soil salt solute spatial distribution in the Yellow River delta. *Acta Geographica Sinica*, 60(3): 511-518.

## APPENDIX A

### $\Psi_X$ AND $\Psi_Y$ MATRIX DEFINITION

If we apply equation (11) for the general case where  $l=1, \dots, mx-1$  and  $k=0, \dots, my$ , we obtain:

$$\begin{aligned}
 q_{dx} = & (2d_{1,0}-d_{0,0}-d_{2,0})^2 + (2d_{2,0}-d_{1,0}-d_{3,0})^2 + \dots + (2d_{mx-1,0}-d_{mx-2,0}-d_{mx,0})^2 + \\
 & + (2d_{1,1}-d_{0,1}-d_{2,1})^2 + (2d_{2,1}-d_{1,1}-d_{3,1})^2 + \dots + (2d_{mx-1,1}-d_{mx-2,1}-d_{mx,1})^2 + \\
 & + \vdots \\
 & + (2d_{1,my}-d_{0,my}-d_{2,my})^2 + (2d_{2,my}-d_{1,my}-d_{3,my})^2 + \dots + (2d_{mx-1,my}-d_{mx-2,my}-d_{mx,my})^2
 \end{aligned} \tag{A1}$$

which can easily be expressed in matrix form as follows:

$$q_{dx} = (\Psi_x \mathbf{d})^T (\Psi_x \mathbf{d}) = \mathbf{d}^T \Psi_x^T \Psi_x \mathbf{d} \tag{A2}$$

where  $\Psi_x \mathbf{d}$  is a vector of  $(m-1)$  elements and has the form:

$$\Psi_x \mathbf{d} = \begin{bmatrix} 2d_{1,0}-d_{0,0}-d_{2,0} \\ 2d_{2,0}-d_{1,0}-d_{3,0} \\ \vdots \\ 2d_{mx-1,0}-d_{mx-2,0}-d_{mx,0} \\ \\ 2d_{1,1}-d_{0,1}-d_{2,1} \\ 2d_{2,1}-d_{1,1}-d_{3,1} \\ \vdots \\ 2d_{mx-1,1}-d_{mx-2,1}-d_{mx,1} \\ \\ 2d_{1,my}-d_{0,my}-d_{2,my} \\ 2d_{2,my}-d_{1,my}-d_{3,my} \\ \vdots \\ 2d_{mx-1,my}-d_{mx-2,my}-d_{mx,my} \end{bmatrix} \tag{A3}$$

From equation (A3) can easily be derived that  $\Psi_x$  is a matrix with size  $(m-1) \times (m+1)$  (for  $i=1, \dots, m-1$  and  $j=0, \dots, m$ ) and  $ij$ th entry:

$$\psi_{x,i,j} = \begin{cases} 2, & \text{when } i=j \text{ and } i-k(mx+1) \notin \{1, mx+1\} \\ -1, & \text{when } |i-j|=1 \text{ and } i-k(mx+1) \notin \{1, mx+1\} \\ 0, & \text{otherwise} \end{cases} \tag{A4}$$

where  $k = 0, \dots, my$ .

By following an equivalent procedure to the above presented, we concluded to the following expression for the smoothness of the bilinear surface according to  $y$  direction:

$$q_{dy} = (\Psi_y \mathbf{d})^T (\Psi_y \mathbf{d}) = \mathbf{d}^T \Psi_y^T \Psi_y \mathbf{d} \quad (\text{A5})$$

where  $\Psi_y$  is a matrix with size  $(m-1) \times (m+1)$  (for  $i=1, \dots, m-1$  and  $j=0, \dots, m$ ) and  $ij$ th entry:

$$\psi_{y,i,j} = \begin{cases} 2, & \text{when } i=j \text{ and } i-l(my+1) \notin \{1, my+1\} \\ -1, & \text{when } |i-j|=1 \text{ and } i-l(my+1) \notin \{1, my+1\} \\ 0, & \text{otherwise} \end{cases} \quad (\text{A6})$$

where  $l = 0, \dots, mx$ . We note that matrices  $\Psi_x$  and  $\Psi_y$  are identical when  $mx = my$ .