Mathematical study of the concept of equivalent reservoir of a reservoir system

Internal Report

Demetris Koutsoyiannis and Athanasia Economou

Department of Water Resources, Faculty of Civil Engineering,

National Technical University, Athens

Heroon Polytechneiou 5, GR-157 80 Zographou, Greece

Abstract It is shown that the performance of a system of two reservoirs is non-superior to that of a hypothetical equivalent reservoir, whose storage capacity, inflows and losses equal the respective sums of those of the separate reservoirs, if the objective is the maximization of release or reliability.

1. Introduction

We consider a system of two reservoirs with storage capacities k^j, $j = 1, 2$, given storages at time $t - 1$ denoted s^j_{t-1} and given demand at time t denoted d^j_t. Assuming that the inflow of reservoir j at time t is a random variable, I^j_t, the release R^j_t, spill W^j_t, and storage S^j_t of the reservoir j at time t will be random variables too. (Here the typical notational convention of upper case letters for random variables is used whereas lower case letters denote known quantities). The release does not necessarily equal the given demand. The reservoir dynamics can be written as

$$S^j_t + R^j_t = s^j_{t-1} - l^j(s^j_{t-1}) + I^j_t - W^j_t \tag{1}$$

where it was assumed that the time step is small, so that the leakage l^j can be assumed a function of the given s^j_{t-1}. (In this theoretical analysis the time step need not equal that used in simulations). Taking expected values in (1), conditional on the previous storage s^j_{t-1}, the storage capacity k^j and the demand d^j_t we obtain

$$E[S^j_t|s^j_{t-1}, k^j, d^j_t] + E[R^j_t|s^j_{t-1}, k^j, d^j_t] = s^j_{t-1} - l^j(s^j_{t-1}) + E[I^j_t] - E[W^j_t|s^j_{t-1}, k^j, d^j_t] \tag{2}$$

where, as implied by the notation, the inflow I^j_t does not depend on s^j_{t-1}, k^j, d^j_t. If the maximization of reliable release or reliability is the management objective, the left hand side of (2) can be regarded a measure of performance of the reservoir operation. Increasing expected release ($E[R^j_t|s^j_{t-1}, k^j, d^j_t]$) results in increasing yield, and increasing expected storage ($E[S^j_t|s^j_{t-1}, k^j, d^j_t]$) results in increasing reliability as it will be more likely to meet demand in subsequent steps. As implied by (2), the expected spill ($E[W^j_t|s^j_{t-1}, k^j, d^j_t]$) can be regarded as an equivalent measure of performance demanding its value to be as small as possible in order for the left hand side of (2) to be as large as possible.

The spill is

$$W_t^j = \max(0, s_{t-1}^j - l^j(s_{t-1}^j) + I_t^j - d_t^j - k^j) \tag{3}$$

or

$$W_t^j = \max(0, I_t^j - y_t^j) \tag{4}$$

where y_t^j encompasses all quantities that are not random variables, i.e.

$$y_t^j := k^j - s_{t-1}^j + l^j(s_{t-1}^j) + d_t^j \tag{5}$$

and represents the empty reservoir space at time t in the case that the inflow I_t^j is zero. If we denote the probability density function and the distribution function of I_t^j as $f_t^j(\ )$ and $F_t^j(\ )$, respectively, from (4) we find

$$E[W_t^j | s_{t-1}^j, k^j, d_t^j] \equiv E[W_t^j | y_t^j] = \int_{y_t^j}^{\infty} (x - y_t^j) f_t^j(x)\, dx \tag{6}$$

2. Performance measure of the equivalent reservoir against the individual reservoirs

Now let us consider the equivalent reservoir as the hypothetical reservoir with capacity $k = k^1 + k^2$, inflow $I_t = I_t^1 + I_t^2$, demand $d_t = d_t^1 + d_t^2$, storage (at time $t - 1$) $s_{t-1} = s_{t-1}^1 + s_{t-1}^2$ and leakage $l(s_{t-1}) = l^1(s_{t-1}^1) + l^2(s_{t-1}^2)$. For the latter to apply for any values of s_{t-1}^1 and s_{t-1}^2, both $l^1(s_{t-1}^1)$ and $l^2(s_{t-1}^2)$ must be linear functions (as implied by the equation $l(s_{t-1}^1 + s_{t-1}^2) = l^1(s_{t-1}^1) + l^2(s_{t-1}^2)$). In a manner analogous to (5) we define y_t for the equivalent reservoir, which will be

$$y_t = y_t^1 + y_t^2 \tag{7}$$

The equivalent reservoir will be non-inferior in performance against the system of the two reservoirs if the quantity

$$g(y, y^1) := E[W^1 | y^1] + E[W^2 | y - y^1] - E[W | y] \tag{8}$$

is non-negative for any y and y^1. Here for simplicity of notation we have omitted the subscript t from all variables. The non-negative value of the 'performance measure' $g(y, y^1)$ implies that the expected spill from the system of two reservoirs is greater than the expected spill from the equivalent reservoir. By virtue of (6), (8) results in

$$g(y, y^1) = \int_{y^1}^{\infty} (x - y^1) f^1(x)\, dx + \int_{y - y^1}^{\infty} (x + y^1 - y) f^2(x)\, dx - \int_{y}^{\infty} (x - y) f(x)\, dx \tag{9}$$

3. Proof of the non-inferior performance of the equivalent reservoir against the individual reservoirs

To show that $g(y, y^1)$ is non-negative, it suffices to show that its minimum value is non-negative. To locate the value of y^1 that minimizes $g(y, y^1)$ for a given y we determine the first derivative with respect to y^1, which after manipulations is found to be

$$\frac{\partial g}{\partial y^1} = F^1(y^1) - F^2(y - y^1) \tag{10}$$

Thus, equating it to zero we can locate y_*^1 for given y from

$$F^1(y_*^1) = F^2(y_*^2), \quad y_*^2 = y - y_*^1 \tag{11}$$

which is a point of minimum since the second derivative

$$\frac{\partial^2 g}{\partial (y^1)^2} = f^1(y_*^1) + f^2(y - y_*^1) \tag{12}$$

is obviously non-negative. It is noted that (11) determines a unique y_*^1 because $F^1(y^1)$ and $F^2(y - y^1)$ are both monotonically increasing and decreasing, respectively, functions of y^1. It can be observed that (11) defines a generalized New-York-City operating rule [*Clark*, 1950] as $F^1(y^1)$ and $F^2(y^2)$ represent the probability of spill for each of the two reservoirs.

Having defined y_*^1 and consequently y_*^2 as functions of y, we can now study the variation of $G(y) := g(y, y_*^1(y))$ with respect to y. We rewrite (9) as

$$G(y) = \int_{y_*^1}^{\infty} (x - y_*^1) f^1(x)\, dx + \int_{y_*^2}^{\infty} (x - y_*^2) f^2(x)\, dx - \int_{y}^{\infty} (x - y) f(x)\, dx \tag{13}$$

and take the derivative, which after manipulations is found to be

$$\frac{dG}{dy} = [F^1(y_*^1) - 1]\frac{dy_*^1}{dy} + [F^2(y_*^2) - 1]\frac{dy_*^2}{dy} - [F(y) - 1] \tag{14}$$

Using (11) and also considering that

$$\frac{dy_*^1}{dy} = 1 - \frac{dy_*^2}{dy} \tag{15}$$

we are able to simplify (14) in the form

$$\frac{dG}{dy} = F^1(y_*^1) - F(y) \tag{16}$$

Equating this derivative to zero and also considering (11), we are able to locate a stationary point y_* from

$$F^1(y_*^1) = F^2(y_*^2) = F(y_*), \quad y_*^1 + y_*^2 = y_* \tag{17}$$

This, however, is not a point of minimum but one of maximum. To show this we find the second derivative

$$\frac{d^2 G}{dy^2} = f^1(y_*^1)\frac{dy_*^1}{dy} - f(y) \tag{18}$$

Besides, taking derivatives in (11) we obtain

$$f^1(y_*^1)\frac{dy_*^1}{dy} = f^2(y_*^2)\left(1 - \frac{dy_*^2}{dy}\right) \tag{19}$$

so that after algebraic manipulations

$$\frac{d^2 G}{dy^2} = \frac{f^1(y_*^1)\, f^2(y_*^2)}{f^1(y_*^1) + f^2(y_*^2)} - f(y) \tag{20}$$

which is negative at $y = y_*$ because when $F^1(y^1) = F^2(y^2) = F(y)$ the following inequality holds

$$\frac{1}{f^1(y^1)} + \frac{1}{f^2(y^2)} \geq \frac{1}{f(y)} \tag{21}$$

(the proof is omitted). We note that the special case of equality in (21) holds only if the inflows to the two reservoirs are dependent in a deterministic manner, i.e. $I^2 = a(I^1)$ where

$a(\,)$ is any function. (In this case it is directly obtained that $f^2(y^2) = f^1(y^1) \,/\, a'(y)$, $f^2(y^2) = f^1(y^1) \,/\, [1 + a'(y)]$, where $a'(y)$ is the derivative of $a(y)$, so that it can be verified that (21) becomes equality). This however, is a case without interest.

Given that the located point y_* is necessarily a point of maximum, this also proves that this point is unique because there cannot be two consecutive stationary points that are points of maximum simultaneously. Consequently, the derivative dG/dy will be positive for $y < y_*$ and negative for $y > y_*$, which means that G is increasing for $y < y_*$ and decreasing for $y > y_*$. Thus, the points of minimum are necessarily the lowest and highest possible values of y. Even in an bi-infinite reservoir, in which case the minimum points are located at $y = \pm\infty$, the value of $G(y)$ cannot be negative. Indeed, from (13) we directly obtain that when $y \to +\infty$, $G(y) \to 0$, whereas for $y \to -\infty$, $G(y) \to y_*^1 + y_*^2 - y_*$, which due to (11) is again zero.

In conclusion, the minimum value of g and G cannot be negative, or equivalently, the expected spill from the equivalent reservoir $E[W|y]$ cannot exceed the sum of the expected spills of the two reservoirs $E[W^1|y^1] + E[W^2|\, y^2]$, for any combination of y^1, y^2, and y. For the proof we did not make any assumption for the distribution of inflows.

4. Example for normally distributed inflows

It may be useful for verification to demonstrate the above theoretical result using a specific distribution. Thus, we assume that inflows to reservoirs 1 and 2 are normally distributed with means μ^1 and μ^2, and standard deviations σ^1 and σ^2. In this case the inflow to the equivalent reservoir, the sum of the inflows to the two reservoirs will be normally distributed too with mean $\mu = \mu^1 + \mu^2$ and standard deviation σ such that $|\sigma^1 - \sigma^2| \le \sigma \le \sigma^1 + \sigma^2$. The probability density function of the latter will be

$$f(x) = \frac{1}{\sqrt{2\,\pi}\,\sigma} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right] \tag{22}$$

so that from (6), after manipulations, we find

$$E[W|y] = \int_y^\infty (x-y) f(x)\, dx = (\sigma)^2 f(y) - (y-\mu)\,[1 - F(y)] \tag{23}$$

Consequently,

$$\begin{aligned} g(y, y^1) = {} & (\sigma^1)^2 f^1(y^1) + (\sigma^2)^2 f^2(y - y^1) - (\sigma)^2 f(y) \\ & - (y^1 - \mu^1)\,[1 - F^1(y^1)] - (y - y^1 - \mu^2)\,[1 - F^2(y - y^1)] + (y - \mu)\,[1 - F(y)] \end{aligned} \tag{24}$$

An example plot of $g(y, y^1)$ versus y^1 for $y = \mu$ is given in Figure 1 (left) assuming certain values of parameters shown in the figure caption. From (11) we conclude that the point of minimum y_*^1 is given by

$$\frac{y_*^1 - \mu^1}{\sigma^1} = \frac{y_*^2 - \mu^2}{\sigma^2}, \quad y_*^2 = y - y_*^1 \tag{25}$$

or

$$y_*^1 = \frac{\sigma^1 (y - \mu^2) + \sigma^2\, \mu^1}{\sigma^1 + \sigma^2} = y - y_*^2 \tag{26}$$

In our example this is $y_*^1 = \mu^1 = 2.48$ hm^3, and the minimum value of $g(y, y^1)$ is 0.14 hm^3.

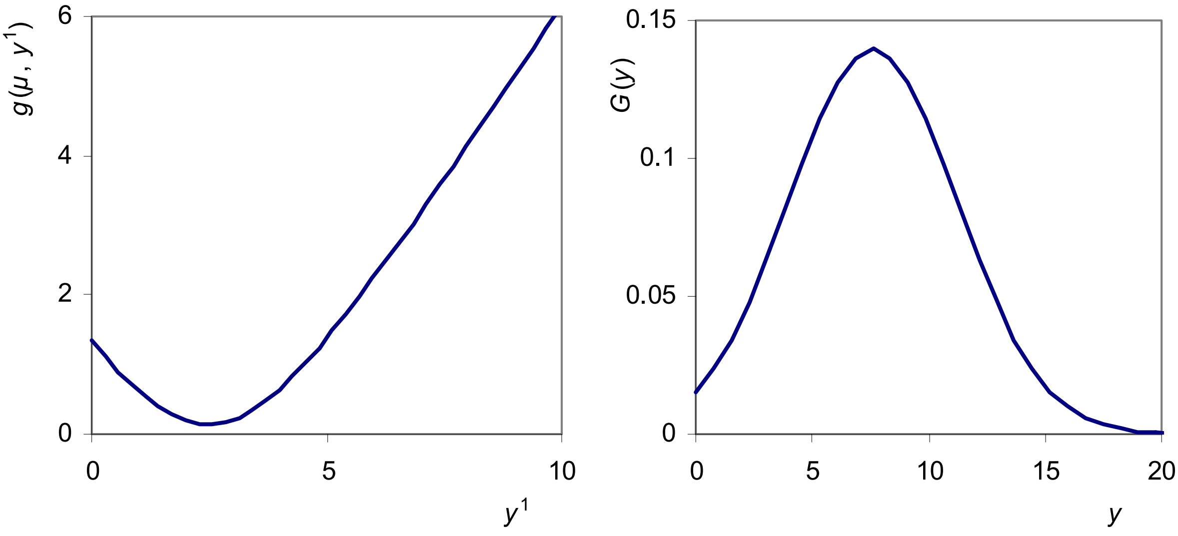

Figure 1 Plots of $g(y, y^1)$ versus y^1 for $y = \mu$ (left) and of $G(y)$ versus y (right) assuming normal distribution of inflows with $\mu^1 = 2.48$, $\mu^2 = 5.13$, $\mu = 7.60$, $\sigma^1 = 1.24$, $\sigma^2 = 2.56$, $\sigma = 3.45$ (corresponding to cross-correlation coefficient 0.60). Units in hm^3.

Now, the function $G(y)$ is

$$G(y) = \sigma^1 (\sigma^1 + \sigma^2) f^1(y_*^1) - (\sigma)^2 f(y) - (y - \mu) [F^1(y_*^1) - F(y)] \tag{27}$$

A plot of $G(y)$ versus y for our example is given in Figure 1(right). The point of maximum is $y_* = \mu$ in which case $y_*^1 = \mu^1$ and $y_*^2 = \mu^2$ and the maximum value of $G(y)$ is 0.14 hm^3. $G(y)$ tends to 0 as y tends to $\pm\infty$.

5. Example for gamma distributed inflows

The normal distribution is not representative if the time step is small, because inflows at small time steps are asymmetric and typically are generated from a three-parameter gamma distribution. The demonstration of the above results with the gamma distribution is not as simple and accurate as in the normal case, mainly because the sum of two correlated, gamma distributed variables (inflows to individual reservoirs) is difficult to determine in an analytical manner. We can, however, assume that it is approximately gamma distributed, i.e., with density $F(x) = \Phi(x - c; \kappa, \lambda)$ where c, κ, λ are the location, shape and scale parameters of the distribution, and

$$\Phi(x, \kappa, \lambda) := \int_0^x \frac{\lambda^\kappa \xi^{\kappa - 1} e^{-\lambda\xi}}{\Gamma(\kappa)} d\xi \tag{28}$$

From (6), after manipulations, we find

$$E[W|y] = \int_y^\infty (x - y) f(x)\, dx = \frac{\kappa}{\lambda} [1 - \Phi(y - c; \kappa + 1, \lambda)] - (y - c) [1 - \Phi(y - c; \kappa, \lambda)] \tag{29}$$

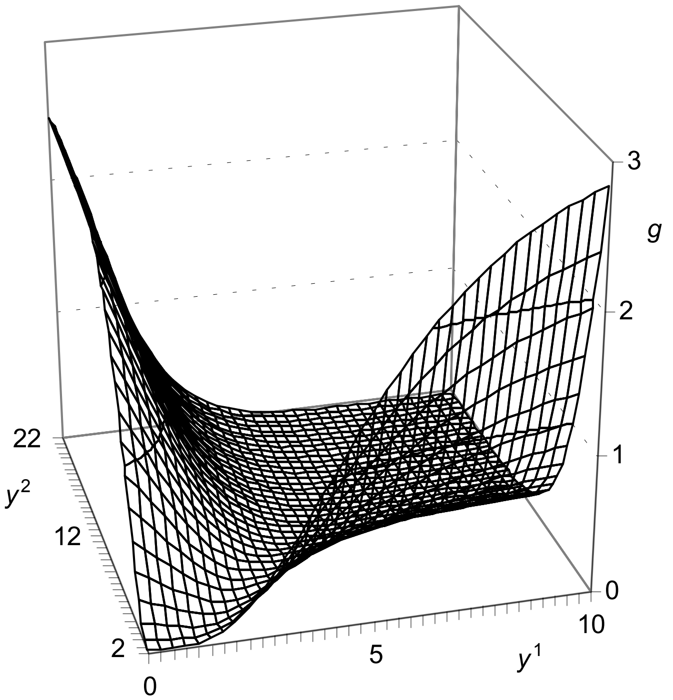

Figure 2 Plots of $g(y, y^1)$ versus y^1 and $y^2 = y - y^1$ assuming three-parameter gamma distribution of inflows with means and standard deviations as in Figure 1 and skewness coefficients $C_s^1 = 1.0$, $C_s^2 = 1.50$, $C_s = 1.05$. Units in hm^3.

Consequently,

$$\begin{aligned} g(y, y^1) = {} & \frac{\kappa^1}{\lambda^1} [1 - \Phi(y^1 - c^1; \kappa^1 + 1, \lambda^1)] + \frac{\kappa^2}{\lambda^2} [1 - \Phi(y - y^1 - c^2; \kappa^2 + 1, \lambda^2)] \\ & - \frac{\kappa}{\lambda} [1 - \Phi(y - c; \kappa + 1, \lambda)] - (y^1 - c^1) [1 - \Phi(y^1 - c^1; \kappa^1, \lambda^1)] \\ & - (y - y^1 - c^2) [1 - \Phi(y - y^1 - c^2; \kappa^2, \lambda^2)] + (y - c) [1 - \Phi(y - c; \kappa, \lambda)] \end{aligned} \tag{30}$$

A 3-dimensional example plot of $g(y, y^1)$ versus y^1 and y^2 ($= y - y^1$) is given in Figure 2. The behavior shown in Figure 2 is similar to that in Figure 1. The performance measure g is non-negative everywhere and becomes zero for very small or very high values of both y^1 and y^2 but it takes high values if y^1 is low and y^2 high or the reverse. This demonstrates that the equivalent reservoir is always non-inferior to the individual reservoirs. For intermediate values of y^1 and y^2, if a good operation rule is followed (approximately, along the diagonal of Figure 2 passing from the origin) the equivalent reservoir is only slightly superior to the individual reservoirs. But if a bad operation is followed, the loss for the two reservoirs can be as high as about 3 hm^3, which is 40% of the total expected inflow of the month.