Medium-range flow prediction for the Nile: a comparison of stochastic and deterministic methods

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Abstract Due to its great importance, the availability of long flow records, contemporary as well as older, and the additional historical information of its behaviour, the Nile is an ideal test case for identifying and understanding hydrological behaviours, and for model development. Such behaviours include the long-term persistence, which historically has motivated the discovery of the Hurst phenomenon and has put into question classical statistical results and typical stochastic models. Based on the empirical evidence from the exploration of the Nile flows and on the theoretical insights provided by the principle of maximum entropy, a concept newly employed in hydrological stochastic modelling, an advanced yet simple stochastic methodology is developed. The approach is focused on the prediction of the Nile flow a month ahead, but the methodology is general and can be applied to any type of stochastic prediction. The stochastic methodology is also compared with deterministic approaches, specifically an analogue (local nonlinear chaotic) model and a connectionist (artificial neural network) model based on the same flow record. All models have good performance with the stochastic model outperforming in prediction skills and the analogue model in simplicity. In addition, the stochastic model has other elements of superiority such as the ability to provide long-term simulations and to improve understanding of natural behaviours.

Key words stochastic *vs* deterministic models; artificial neural networks; linearity and nonlinearity; maximum entropy; Hurst phenomenon; Nile

Prévision du débit du Nil à moyen terme: une comparaison de méthodes stochastiques et déterministes

Résumé En raison de son importance, de la disponibilité de longues séries de données de débit, contemporaines et anciennes, et d'informations historiques additionnelles sur son comportement, le Nil est un cas idéal pour identifier et comprendre les comportements hydrologiques, et pour développer des modèles. Les comportements en question incluent la persistance à long terme, qui est historiquement à l'origine de la découverte du phénomène de Hurst, et qui a remis en question des résultats statistiques classiques et des modèles stochastiques typiques. Une méthodologie stochastique avancée et néanmoins simple est développée, sur la base d'une part d'observations empiriques permises par l'exploration des données de débit du Nil et d'autre part de développements théoriques permis par le principe de l'entropie maximale, un concept nouvellement utilisé en modélisation hydrologique stochastique. L'approche est focalisée sur la prévision du débit du Nil à échéance de un mois, mais la méthodologie est générale et peut être appliquée à tout type de prévision stochastique. La méthodologie stochastique est également comparée avec des approches déterministes, en particulier un modèle analogue (chaotique non-linéaire local) et un modèle connectioniste (réseau de neurones artificiels) basés sur les mêmes données de débit. Tous les modèles présentent de bonnes performances, les modèles stochastique et analogue étant meilleurs respectivement en qualité de prévision et en simplicité. De plus, le modèle stochastique présente d'autres éléments de supériorité comme les aptitudes à fournir des simulations à long terme et à améliorer la compréhension des comportements naturels.

Mots clefs modèles stochastique *vs* déterministe; réseaux de neurones artificiels; linéarité et non-linéarité; entropie maximale; phénomène de Hurst; Nile

1 INTRODUCTION

We predict, God laughs (Paraphrase of an old proverb)

The use of stochastic models in hydrological tasks such as simulation and prediction has a history of half a century, since the pioneering works of Barnes (1954) and Thomas & Fiering (1962). The classical book on time series analysis, forecasting and control by Box & Jenkins (1970) has greatly influenced stochastic hydrology. The model classes it proposed (autoregressive—AR; moving average—MA and combinations of the two—ARMA) have become classical and are still very popular.

However, these model types are not ideal for hydrological processes for several reasons. First, all Box-Jenkins models are essentially of short-range dependence (SRD), that is, their auto-correlation structure decays exponentially with lag time; in contrast, there is evidence that hydrological processes exhibit long-range dependence (LRD), i.e. power-type decay of auto-correlation also known as the Hurst phenomenon (e.g. Koutsoyiannis, 2005b). Second, these models rely largely on a normality assumption, whereas it is known that hydrological processes (mostly on sub-annual scales) depart from normality and perhaps have distribution tails of power-type (e.g. Koutsoyiannis, 2005a). Third, the seasonal behaviour exhibited by hydrological processes at sub-annual scales is complex (with distribution type and dependence structure that change within the year) and cannot be handled by "deseasonalization" techniques typically used in companion to Box-Jenkins models (Koutsoyiannis & Georgakakos, 2006). Fourth, except for simplified processes, such as AR(1) or ARMA(1,1), the models are not parsimonious as they involve many parameters estimated from the data. In fact, the structure of these models is tightly linked to the number of parameters and one cannot change the structure (e.g. increase the tail of the dependence) without using additional parameters. However, typical statistical samples do not allow a reliable estimation of many parameters. This is particularly the case for processes with LRD, as this behaviour entails parameter uncertainty dramatically higher than in SRD (Koutsoyiannis, 2003; Cohn & Lins, 2005, Koutsoyiannis & Montanari, 2007). Fifth, again except for low order processes (i.e. AR(1) and ARMA(1,1)), other processes of this family do not have a physical meaning and thus are used in a rather black-box setting.

Several of the above drawbacks have been remedied by adaptations or even introduction of different model types. Thus, LRD can be reproduced by specialized models such as fractional Gaussian noise processes (Mandelbrot, 1965) and fractionally differenced ARMA processes (Hosking, 1984), or by generalized generation schemes applied on generalized autocorrelation structures (Koutsoyiannis, 2000). The latter technique also tackles some of the other problems listed above as it is parametrically parsimonious (i.e. the generation scheme is not tied to the autocorrelation structure), it can handle non-normal distributions and is also multivariate. The seasonality problem has been tackled by cyclostationary (periodic) processes. However, these are necessarily SRD because only low-order processes (such as periodic AR(1) or periodic ARMA(1,1); Bras & Rodriguez-Iturbe, 1985; Koutsoyiannis, 1999) are computationally feasible in a cyclostationary setting. Yet, however, in simulation mode (as opposed to prediction mode) the stochastic generation problem has been tackled using a disaggregation logic, in which an annual time series is generated first and then the annual amounts are disaggregated into seasonal. Thus, the LRD properties are handled by an appropriate stationary model such as those stated above, the seasonality is handled by a cyclostationary model with SRD, and the two models are coupled so that the latter becomes operationally consistent with the former (Koutsoyiannis, 2001). An approach that can generate cyclostationary time series with LRD without disaggregation was proposed recently (Langousis & Koutsoyiannis, 2006), but this is again for simulation.

Thus, while for stochastic simulation there exist advanced techniques (generally departing from ARMA model types) that are consistent with the peculiarities of hydrological processes, this does not happen in stochastic prediction. The modelling techniques for simulation may not be directly adaptable for prediction in a cyclostationary setting. For instance, a disaggregation framework is not appropriate for prediction.

In the last decade, this gap has been covered by techniques structurally different from stochastic techniques. These are based on recent advances on nonlinear dynamical systems (or chaotic systems) and have a deterministic basis. Most popular among these deterministic model categories are the "analogue" models also called "local nonlinear chaotic" models, and the "connectionist" models or metaphorically "artificial neural network" models. The application of such models in hydrological prediction are numerous in recent years (a sample from the last few years includes Tomasino *et al.*, 2004; Hu *et al*., 2005; Kisi, 2005; Giustolisi & Laucelli, 2005; Shouyu & Honglan, 2005; Giustolisi & Simeone, 2006; Jayawardena *et al*., 2006; Abrahart *et al*., 2007; Corzo & Solomatine, 2007; de Vos & Rientjes, 2007; Muluye & Coulibaly, 2007; see also

See *et al*., 2007) and their performances are impressive in most cases. A major weak point of these model types is that, despite being deterministic in conception, in most cases they are data driven and black box, thus providing no process insight and involving no structured reasoning in their formulation, except in few simple model types (e.g. Solomatine & Dulal, 2003; Giustolisi & Savic, 2006). Another weak point is the fact that they do not provide tools for Monte Carlo simulation (this is demonstrated below).

The nonlinear and data-driven character of these models has inspired many to devise stochastic models with such features. Such models (pioneered by Lall & Sharma, 1996) can implement a nonlinear data-driven (as opposed to linear parametric) dependence structure of the process and can reproduce the historical histogram in lieu of a theoretical distribution function. In our opinion, however, these may be weak rather than strong points of these model types. As we will discuss below, linearity has a totally different meaning in deterministic and stochastic approaches and is justifiable in a stochastic framework. Also, the use of theoretical models justified by reasoning, is a powerful and insightful feature in stochastic modelling and should not be replaced by high uncertainty estimates of merely empirical basis, which after all are not appropriate to perform extrapolations that are largely needed. For instance, asymptotic probabilistic laws such as the Law of Large Numbers and the Central Limit Theorem provide the theoretical basis for estimation and prediction, when the conditions for their applicability are satisfied. The principle of maximum entropy (ME), is the most powerful theoretical tool of this type. Essentially, it is a probabilistic law, but simultaneously is a physical principle as strong as to provide the formulation of the second law of thermodynamics. As a first principle in physics, it is quite different from other first principles such as Newton's laws, because of the probabilistic character of the entropy concept (e.g. Stowe, 2007). As demonstrated recently (Koutsoyiannis, 2005a,b), the ME principle can explain several macroscopic behaviours of hydrological processes.

In this paper we propose a general stochastic framework which is in full agreement with the features of hydrological process and the requirements for prediction, i.e. (a) it admits and utilizes LRD; (b) it can perform with distributions of either exponential or hyper-exponential tails; (c) it is cyclostationary; (d) it is parameter parsimonious; (e) it is insightful as it has a strong theoretical basis for inference (principle of ME); (f) it can perform in both prediction and simulation; and (g) it is simple and easily applicable. In addition, we compare this stochastic approach with two data-driven models following a deterministic logic, namely an analogue model and a connectionist model. The comparison includes both theoretical issues as well as the prediction skills as derived from a test case study.

2 STUDY LOCATION AND DATA

The test case pertains to the Nile River and has significant interest both from a research as well as an operational standpoint. The Nile River is spread over 10 East African countries with numerous water uses, including water supply for domestic, industrial, and agricultural use, energy generation, flood protection, and environmental management, among others (Georgakakos, 2007). Medium- and long-range Nile flow prediction is critical for the operation of the existing water control projects from Lake Victoria (in Kenya, Tanzania, and Uganda) to the High Aswan Dam in Egypt, and several planned facilities in the middle reaches (Ethiopia and Sudan). The forecast lead time considered in the case study and model intercomparison is one month; however, the framework is general enough and can perform in longer lead times albeit with lower skill. The Nile is the world's longest river (6670 km) with water travel times that, depending on the season, vary from 20 days (Blue Nile tributary) to more than 45 days (White Nile tributary). This induces strong dependence on a monthly time scale and, along with other storage mechanisms in the catchment, makes monthly forecast feasible by using as explanatory (predictor) variables merely past Nile flows at the same site where prediction is made. Obviously additional explanatory variables (e.g. rainfall or river flow at an upstream site) would enhance the prediction capacity but this is out of the scope of this work, which does not make use of additional explanatory variables.

However, the methodology proposed is straightforward to apply to cases of multiple predictor variables, because by construction it is a multivariate methodology.

The modern flow record at Aswan is one of the longest worldwide (131 years) and makes analysis and modelling reliable. In addition, older instrumental records of annual maximum and minimum water level at the Roda Nilometer for more than 800 years also exist. All flow records as well as additional historical and archaeological data (Said, 1993) affirm the LRD behaviour of the Nile flows and raise the demand that this dependence should be incorporated in a stochastic model, either for simulation or prediction. Conceptually, LRD is nothing other than multi-scale variability or multi-scale fluctuation (Koutsoyiannis, 2002), and greatly increases uncertainty of statistical estimation and long-term prediction (Koutsoyiannis, 2006). One may argue that deterministic climatic models, if used instead of LRD stochastic models, may result in reduced uncertainty in lieu of the high uncertainty implied by LRD. However, to our knowledge, no coupled climatic-hydrological model has yet explained the observed past high multi-scale variability of the Nile. The problem is then: how could one be confident about future projections of a model? In addition, as Conway (2005) concludes, analysis of climate change projections for the region shows that there is no clear indication of how Nile flows will evolve in the future because of uncertainty about future rainfall patterns in the basin.

All prediction models in this study are based on the available time series of Nile flows, which is 131 years long. For the stochastic model, the record is divided into two periods, a 78-year period for model fitting (60% of the total length; from hydrologic year 1870/71 to 1947/48) and a 53-year period for model validation (from 1948/49 to 2000/01). For the deterministic data-driven models, the fitting period is further subdivided into two sub-periods, a 52-year calibration period (two thirds of the total, from 1870/71 to 1921/22) and a 26-year verification period (from 1922/23 to 1947/48). In this way, all model fitting procedures are done exclusively within the 78-year period in all cases, thus enabling a fair comparison of all models in the validation period, the data of which were not used in model fitting.

3 MODELLING APPROACHES AND UNDERLYING CONCEPTS

Let us consider a simple stochastic model which attempts to issue a prediction W of the flow in a specified month, say December, based on a single explanatory variable Z, say the flow in the same year in November. In a stochastic approach, W and Z are thought of as random variables (for this reason we have used an upper case convention, whereas for values, e.g. observations of the process, we use lower case letters). W is assumed stochastically dependent on Z. The dependence manifests the stochastic dynamics of the process, which can be represented by:

$$W = g_s(Z, V) \tag{1}$$

where $g_s(\,)$ is an appropriate function and V is an additional random variable (assumed independent of Z) whose involvement manifests the fact that the dependence between the variables W and Z is not fully deterministic. If we know a realization of Z which is an observation z of the natural process, then we can calculate a point prediction of W by $E[W|Z = z] = E[g(z, V)]$, where $E[\,]$ denotes expectation. Using (1), we can also calculate prediction intervals of W for any desired confidence, either by analytical means or by Monte Carlo simulation.

To obtain any type of prediction we need to know the function $g_s(\,)$. We can obtain an idea of the type of this function by constructing a plot of historical observations of W and Z, such as that of Fig. 1 (left panel). This plot reveals a practically (macroscopically) linear arrangement of points and suggests that $W = g_s(Z, V) = a\,Z + V$ (where a is a parameter). In the figure we have plotted an instance of this equation (marked "Stochastic linear model 2") substituting the observation z for the variable Z and the mean $E[V]$ for the variable V. The reader who is not satisfied with the linear relationship and prefers another monotonic relationship (e.g. a power law) may feel free to replace the equation with a new one. Again we can linearize the new equation, provided that it is monotonic, by appropriate transformation of the variables. In fact, we have already done it and we

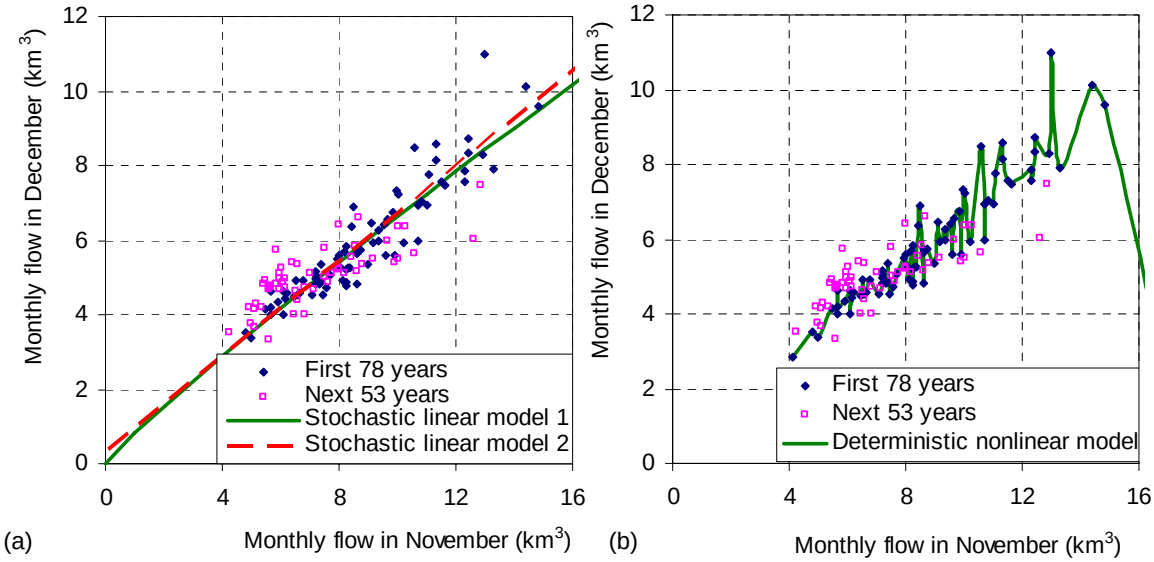

Fig. 1 Graphical depiction of the basic concept of (a) a stochastic and (b) a deterministic approach in the hypothetical case of a model with a single time delay component. Stochastic linear models 1 and 2 are models assuming linear dependence of the normalized and natural flows, respectively. Deterministic linear model is an arbitrary hypothetical non self-intersecting curve passing through all 78 points.

have plotted in Fig. 1(a) another model, resulting from a transformation that will be discussed later. We call this second model "Stochastic linear model 1" because it is still linear for transformed variables.

The fact that in this case the stochastic relationship appears to be so simple (linear) should not be regarded as a fortunate coincidence. Rather it seems to be the rule in hydrological and other processes and this does not contradict a commonly accepted statement that natural processes are nonlinear. Both statements are correct: nonlinearity is absolutely necessary when one uses a deterministic description of processes, but linearity is the rule when one uses a stochastic description. It has been argued that a heavily nonlinear system may become approximately linear again (Penland, 2006). This reflects the fact that when studying the detailed dynamics of a complex system using a deterministic description, then this dynamics would most likely be highly nonlinear. However, when we study the macroscopic behaviour of a highly complex system based on its trajectory, using (because of high complexity but perhaps inadvertently) a stochastic description, then linearity may emerge. An explanation for this emergence of linearity could be traced on the principle of ME. In particular, it is known (Papoulis, 1991) that maximization of the standard joint entropy (see definition in Section 4.1) results in a multivariate normal distribution. This entails a linear dependence of the lagged flows W and Z (or transformation thereof, if generalized definitions of entropy are used; see Section 4.1).

In a deterministic approach the system dynamics is fundamentally different and, particularly, the concept of linearity has a meaning completely different from that in a stochastic approach. According to a deterministic approach, the lagged flows are not random variables and the system dynamics is a deterministic relationship of them, i.e.:

$$w = g_d(z) \tag{2}$$

where $g_d(\,)$ is a function such as the hypothetical (caricature) curve shown in Fig. 1(b) passing through all 78 points of the fitting period. Topologically, this curve should necessarily be non-monotonic (thus nonlinear and non linearizable) and non self-intersecting. A linear function would result in a system trajectory that would be either completely stable (attracted to a specific finite value) or runaway (tending to infinity). (This is the reason why the simplest and most popular function used for demonstration of deterministic dynamical systems is the logistic function rather than a linear function.) If the caricature function of Fig. 1(b) represented the true dynamics of the

process, then the additional 53 points of the validation period would lie on the curve. Since in fact they lie outside the curve, we should either change the function $g_d(\ )$ or add explanatory variables, e.g. additional lagged (delayed) flows, and replace the single variable z in (2) with a vector of variables $\boldsymbol{z} = [z_{(1)}, z_{(2)}, \ldots, z_{(m)}]^T$, where $z_{(i)}$ are the explanatory variables (e.g. the lagged flows). For a sufficient number of variables (known in dynamical systems literature as the *embedding dimension*) m and an appropriate function $g_d(\ )$ we may anticipate reconstructing the dynamics of the system if the system is in reality deterministic. Whilst additional variables could be also used in the stochastic case (1), there is a fundamental difference in the two cases: in the deterministic formulation the purpose of using additional variables is to faithfully reconstruct the system dynamics, whereas in the stochastic formulation the purposes are to better represent the dynamics and to reduce the prediction error, that is to minimize the variance of the random variable V.

According to the analogue (local linear) approach, the prediction is done without explicitly determining the function $g_d(\ )$. Instead, the method tries to locate within the historical record a vector $\boldsymbol{z}^1$ that is nearest (in state) to the "current" vector $\boldsymbol{z}$ (i.e. an analogue past state of the system). The value w^1 next (in time) to $\boldsymbol{z}^1$, which is known from the record, could be used as a prediction for the future of the current system state. In fact, the algorithm uses more than one nearest past state, as will be explained later (Section 5.1).

In contrast, in the connectionist (artificial neural network) approach the function $g_d(\ )$ is determined, although it is usually so complicated that we do not even write its mathematical expression. An understanding of the complex relationships between inputs (function arguments) $\boldsymbol{z}$ and outputs w within a connectionist model is offered by the so called Kolmogorov's (1957) superposition theorem, according to which any continuous real function $g_d(\boldsymbol{z})$ of a vector variable $\boldsymbol{z}$ (defined on the m-dimensional hypercube $[0, 1]^m$) can be represented as a superposition and composition of continuous functions of only one variable. Formally, the theorem says that there exist continuous real functions $h_{ij}(z)$ and $g_i(z)$ such that:

$$g_d(\boldsymbol{z}) = \sum_{i=1}^{2m+1} g_i\left(\sum_{j=1}^{m} h_{ij}(z_j)\right) \tag{3}$$

As shown by Kurkova (1992), based on earlier results by Hecht-Nielsen (1987), and discussed by Beiu & Zawadzki (2005), it is possible to adapt Kolmogorov's theorem to an artificial neural network and to approximate the functions h and g by staircase-like functions.

Connectionist models typically use sigmoidal elementary functions ($\sigma(z) = 1/(1 + e^{bz-c})$) and perform weighted sums and compositions of many of them, according to some rules determined by a geometric analogue of nodes and arcs forming a network. The network topology includes an "input layer" with m nodes, an output layer with one node and one or more "hidden" layers (in the case that (3) applies, there is only one hidden layer with $2m + 1$ nodes). Kurkova (1992) showed that connectionist models with standard sigmoidal functions and only two hidden layers could approximate any continuous function with arbitrary precision, but the number of units needed for a good approximation is exponential on m.

The three models, one stochastic and two deterministic with the above general features, are applied here in several configurations that will be detailed in the following sections. A synopsis of models and configurations is given in Table 1.

4 STOCHASTIC MODEL

Before we can construct a stochastic model it is necessary to study the marginal and dependence properties of the process of interest, here the Nile flow. Its summary statistics on a monthly and annual basis are given in Table 2. The convention of a hydrological year is used, which for the Nile flows at Aswan, Egypt, is assumed to start on 1 August. From Table 2, two different regimes are typically observed. The flood period from August to October, when most of the flow comes from the Blue Nile, and the November to July base flow period when the flow is sustained

Table 1 List of all models of the study.

Model type	Model specifications	Model abbreviation
Stochastic	Cyclostationary with short- and long-range dependence, using normalizing transformation of time series	S1
	As S1 but without normalizing transformation	S2
	PAR(2) without normalizing transformation	S3
Analogue (local linear)	Constant delay, 12 consecutive delay items, 11 neighbours	A1
	Constant delay, 13 consecutive delay items, 24 neighbours	A2
	Variable delay, 4 delay items, 7 neighbours	A3
Connectionist (artificial neural network)	Constant delay, 5 inputs, 2 hidden layers, 2+2 hidden nodes	C1
	Constant delay, 14 inputs, 2 hidden layers, 11+11 hidden nodes	C2
	Variable delay, 4 inputs, 2 hidden layers, 4+2 hidden nodes	C3

Table 2 Main marginal and dependence statistics of the untransformed 78-year record on monthly and annual basis.

Month	μ (km^3)	σ (km^3)	C_s	C_k	τ_3	τ_4	H	ρ_1	ρ_2	ρ_{12}
August	19.37	4.62	–0.09	–0.14	0.00	0.12	0.76	0.71	0.26	0.16
September	22.98	4.29	–0.12	–0.57	–0.02	0.07	0.74	0.80	0.51	0.17
October	16.33	3.65	0.41	0.31	0.08	0.14	0.76	0.88	0.70	0.24
November	8.79	2.34	0.42	–0.27	0.09	0.11	0.80	0.90	0.77	0.26
December	5.92	1.60	0.86	0.60	0.19	0.13	0.89	0.94	0.85	0.42
January	4.37	1.20	0.64	0.31	0.15	0.15	0.88	0.98	0.91	0.44
February	3.02	1.00	0.85	0.27	0.20	0.12	0.82	0.96	0.92	0.35
March	2.51	0.96	1.25	1.34	0.26	0.17	0.78	0.91	0.84	0.31
April	1.89	0.75	1.75	3.56	0.33	0.19	0.78	0.94	0.78	0.33
May	1.68	0.63	2.13	6.30	0.33	0.23	0.72	0.93	0.85	0.30
June	1.91	0.68	1.89	6.00	0.27	0.20	0.63	0.70	0.59	0.11
July	5.06	1.84	0.75	0.24	0.16	0.12	0.89	0.65	0.44	0.47
Average			0.90	1.50	0.17	0.14	0.79	0.86	0.70	0.30
Annual	93.85	20.16	0.35	–0.08	0.09	0.09	0.85	0.35	0.35	

Notation μ: mean; σ: standard deviation; C_s: standard coefficient of skewness; C_k: standard coefficient of kurtosis, τ_3: L-coefficient of skewness; τ_4: L-coefficient of kurtosis, H: Hurst coefficient; ρ_1 and ρ_2: autocorrelation coefficients for lags 1 and 2 (for the monthly series they are autocorrelations of the current month with one or two months before; for the annual series they are autocorrelations of the current year with one or two years before); ρ_{12} (for the monthly series): autocorrelation of current month to the same month one year before.

by the White Nile. The two periods will be referred to as the high-flow and low-flow periods, respectively.

4.1 Marginal distribution

Generally, the selection of a distribution function for use in a stochastic hydrological model is done empirically, based on comparisons of empirical statistics with theoretical ones of a repertoire of common distribution functions. Koutsoyiannis (2005a) proposed that theoretical reasoning could also assist this selection, and pointed to ME as the physical and mathematical principle that can be the basis for such reasoning. For a continuous random variable X with density $f(x)$ the standard entropy, also known as Boltzmann-Gibbs-Shannon entropy, is by definition (e.g. Papoulis, 1991):

$$\varphi := E[-\ln f(X)] = -\int_{-\infty}^{\infty} f(x) \ln f(x)\, \mathrm{d}x \tag{4}$$

A generalization of this definition, fruitfully used in several scientific fields including physics, chemistry, biology, economics, medicine, computer sciences and social sciences, and also useful in hydrology has been offered by Tsallis (1998, 2004):

$$\varphi_q = \frac{1 - \int_{-\infty}^{\infty} [f(x)]^q \, \mathrm{d}x}{q - 1} \tag{5}$$

It can be easily checked that the limit for $q \to 1$ precisely reproduces the Shannon entropy, i.e. $\varphi_1 \equiv \varphi$. The above definitions can be generalized for vectors of random variables, where $f(\boldsymbol{x})$ is the joint density and φ_q or φ are joint entropies.

Maximization of standard entropy (equation (4)) (i.e. application of the ME principle) with simple constraints of known mean μ and variance σ^2 results in (e.g. Papoulis, 1991):

$$f(x) = \exp(-\lambda_0 - \lambda_1 x - \lambda_2 x^2) \tag{6}$$

where λ_0, λ_1 and λ_2 are parameters depending on μ and σ. Inspection of equation (6) shows that it is the normal density function. If the variable under study X is by definition non negative, as is the case for hydrological and most geophysical variables, maximization of entropy should incorporate the additional inequality constraint $x \geq 0$. In this case the resulting ME distribution is given by equation (6) again, but defined on $x \geq 0$, and it is the truncated normal distribution.

As discussed in Koutsoyiannis (2005a), the truncated normal distribution fails to describe cases in which the variation $\sigma/\mu > 1$. To find a ME solution for such cases, one should use Tsallis entropy (also known as nonextensive or non-additive entropy) in lieu of standard entropy. Maximization of Tsallis entropy φ_q in equation (5) with known μ and σ^2 yields a hyper-exponential (power-type distribution), i.e.:

$$f(x) = [1 + \kappa (\lambda_0 + \lambda_1 x + \lambda_2 x^2)]^{-1 - 1/\kappa} \tag{7}$$

where $\kappa := (1-q)/q$ (Koutsoyiannis, 2005). For $\kappa \to 0$ ($q \to 1$), equation (7) switches to equation (6). Equation (7) is mathematically equivalent to the so-called Tsallis distribution (Tsallis *et al*., 1995; Prato & Tsallis, 1999), as can be verified replacing $1-q$ with $q-1$. Note that the latter distribution has been obtained by constraining optimization not with the typical first and second moments as above but with generalized ones, known as q-expectations.

The fact that high variation σ/μ is common in hydrological variables at fine time scales is a strong indication of the applicability of the Tsallis ME principle in hydrology. The most essential difference of equation (7) with respect to equation (6) is the implied hyper-exponential tail of distribution, which is quantified by parameter κ (for $\kappa > 0$). When the process of interest is aggregated from fine to coarser scales, σ/μ becomes smaller and smaller and the hyper-exponential behaviour of the tail becomes less and less visible from data due to the central limit theorem. However, it can be easily shown that theoretically the tail of the distribution is still hyper-exponential with the same κ, although the mathematical form of the distribution (7) is not preserved exactly (in contrast to (6) which is preserved in aggregation). Nonetheless, equation (7) can be used as an approximation over a range of scales.

The above are investigated for the Nile flow as shown in Table 2 and Fig. 2. The latter depicts normal probability plots of the distribution functions of annual and monthly Nile flows for August and December, which are representative of the high-flow and low-flow periods, respectively. The variation σ/μ is 0.20 for the annual flows and ranges from 0.23 to 0.43 for the monthly flows; these values can support the appropriateness of the (truncated) normal distribution. It can be seen that the normal distribution is a satisfactory approximation for the annual flows, as well as for the monthly flows in August. This is also the case for September and October. Clearly, however, the flows in December exhibit a heavier tail, and this is observed for all months in the low flow period. In all these months, the empirical coefficients of skewness are positive (see Table 2) and those of kurtosis (and L-skewness and L-kurtosis) are higher than those of the normal distribution. Therefore, for the nine months of low flows, the departures of the empirical distributions from the normal could be attributed to over-exponential tails and could justify the use of (7) instead of (6).

As the normal distribution is very convenient in building a stochastic model for either simulation or prediction, one can think of applying a normalizing transformation $Z = g(X)$ to the

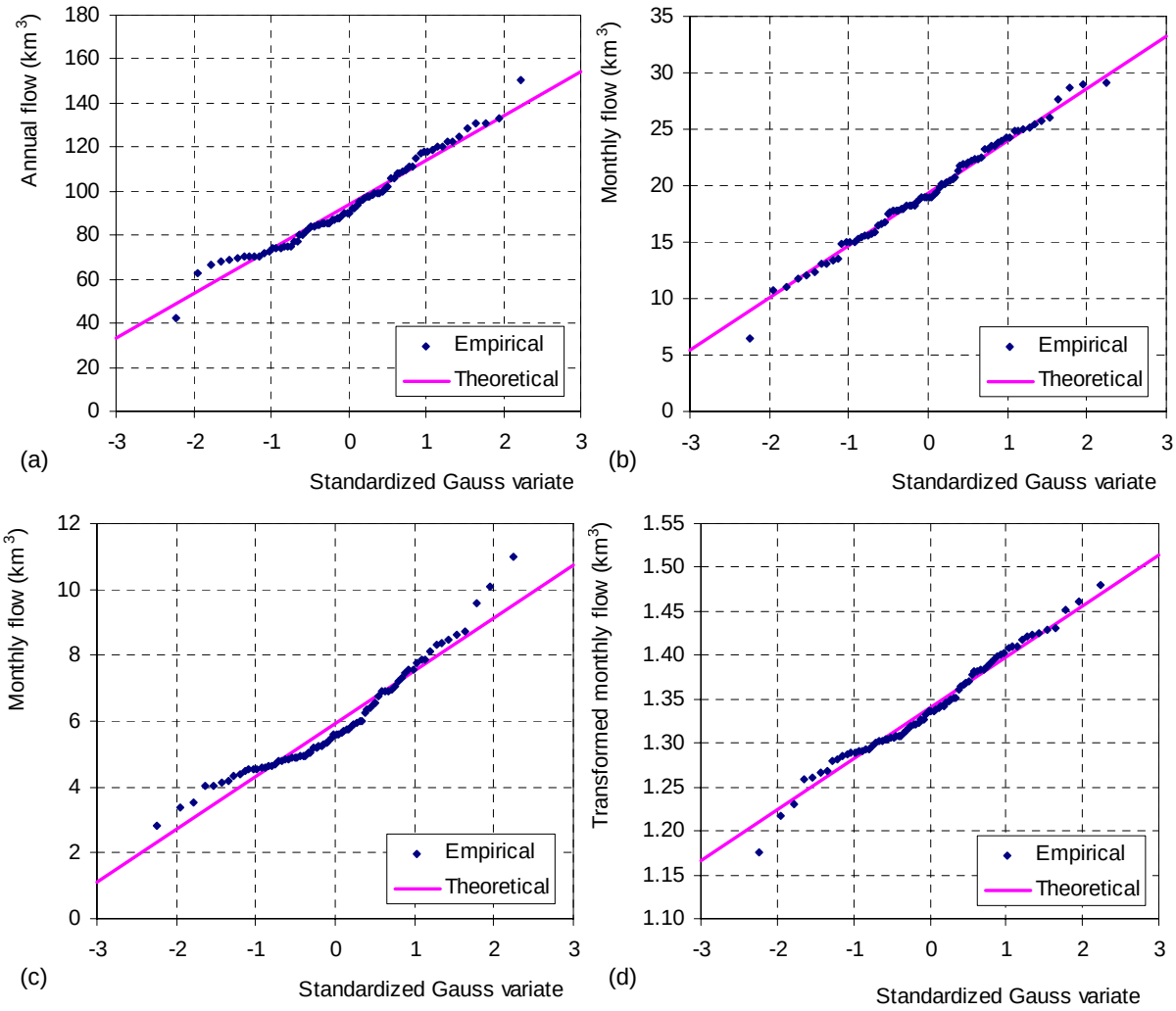

Fig. 2 Normal probability plots of the distribution function (empirical and normal) of annual and monthly Nile flows: (a) annual; (b) August; (c); December; and (d) December, but after applying normalizing transformation (9).

variable of interest X, instead of using the non-normal distribution (7). In this case, Z will have a distribution that is approximated by (6), whereas X has a distribution that is approximated by (7). For appropriate selection of a translation parameter c, based on equations (7) and (6), we can write:

$$\exp[-\lambda_2 (z-c)^2] \sim [1 + \kappa \lambda'_2 (x-c)^2]^{-1-1/\kappa} \quad (8)$$

from which (after algebraic manipulation and change of parameters) we obtain the following normalizing transformation:

$$Z = g(X) - g(0);\ g(x) = c + \mathrm{sgn}(x-c)\,\lambda \sqrt{\left(1+\frac{1}{\kappa}\right) \ln\left[1 + \kappa \left(\frac{x-c}{\lambda}\right)^2\right]} \quad (9)$$

This, in addition to the tail-determining dimensionless parameter κ, contains a scale parameter λ with the same units as x, which enables physical consistency of the transformation, and a translation parameter c, again with the same units as x. It is easily seen that: (a) z has the same units as x; (b) for x/λ ranging in $[0, \infty)$, z/λ also ranges in $[0, \infty)$; and (c) for $\kappa = 0$, z is identical to x.

To apply the normalizing transformation to the monthly Nile flows in the low flow period, one may think of using different parameters κ, λ and c for each month. However, due to the already mentioned high estimation uncertainty, accurate estimation of $9 \times 3 = 27$ parameters is hardly attainable. Therefore, we prefer to assume $c = 0$ and a single pair of parameters κ, λ for all nine

Table 3 Main marginal and dependence statistics of the transformed 78-year record on monthly basis.

Month	μ (km^3)	σ (km^3)	C_s	C_k	τ_3	τ_4	H	ρ_1	ρ_2	ρ_{12}
Aug	19.37	4.623	–0.09	–0.14	0.00	0.12	0.76	0.71	0.25	0.16
Sep	22.98	4.292	–0.12	–0.57	–0.02	0.07	0.74	0.80	0.52	0.17
Oct	16.33	3.649	0.41	0.31	0.08	0.14	0.76	0.88	0.70	0.24
Nov	1.42	0.057	–0.30	–0.16	–0.05	0.11	0.76	0.89	0.77	0.23
Dec	1.34	0.058	0.05	0.07	0.05	0.11	0.88	0.95	0.87	0.40
Jan	1.27	0.065	–0.48	1.26	–0.02	0.17	0.83	0.98	0.92	0.36
Feb	1.18	0.081	–0.13	0.18	0.02	0.11	0.77	0.96	0.93	0.30
Mar	1.13	0.090	0.25	–0.19	0.07	0.12	0.80	0.89	0.81	0.36
Apr	1.05	0.093	0.60	0.08	0.13	0.12	0.84	0.95	0.77	0.44
May	1.02	0.087	0.67	0.72	0.13	0.14	0.77	0.93	0.86	0.36
Jun	1.05	0.086	0.32	0.49	0.07	0.15	0.64	0.69	0.57	0.11
Jul	1.30	0.084	–0.31	0.13	–0.04	0.11	0.90	0.64	0.45	0.50
Average			0.07	0.18	0.03	0.12	0.79	0.86	0.70	0.30

Notation as in Table 2.

months, which we estimate by minimizing the departures of empirical skewness, kurtosis, L-skewness, L-kurtosis of z, aggregated over all months, from those of the normal distribution. The resulting parameters, estimated for the fitting period (78 years) were $\kappa = 2.76$ and $\lambda = 0.47$ km^3. Figure 2(d), which depicts a normal probability plot of the transformed monthly flows z in December, indicates that transformation (9) performed a satisfactory normalization of the distribution. The statistical characteristics of the transformed monthly flows are shown in Table 3.

4.2 Dependence

Given that the marginal distribution of the transformed flows Z is normal, from the discussion of Section 3 it follows that the multivariate distribution of the process Z_i will be multivariate normal, as a result of the ME principle applied on Z_i in a multivariate setting. This entails linear relationships among consecutive variables Z_i. Using the inverse transformation of (9), we can find the relationships between consecutive X_i (this is the case with the curve marked as "Stochastic linear model 1" in Fig. 1(a), but it is more convenient to formulate the entire model in terms of Z_i and use the inverse transformation only in the last phase to translate a prediction for normalized flow Z to a prediction of the actual flow X.

Because of the normality of Z_i, the multivariate distribution can be fully expressed in terms of the autocorrelation function. Table 2 shows that the autocorrelations of the natural monthly Nile flows (corr[X_i, X_{i+j}] for month i = 1 (August) to 12 (July) and lag j = 1, 2, 12) are very high, but differ from month to month. Table 3 shows that the autocorrelations of the normalized flows (ρ_j = corr[Z_i, Z_{i+j}]) have essentially the same values as in the natural flows. Figure 3 depicts the autocorrelograms for two months, August and December, representative for the high- and low-flow periods, respectively, for lags up to 60 (corresponding to five years). Here we can observe that monthly autocorrelations differ significantly from month to month for small lags (periodicity) but become very similar for large lags, for which they keep high values thus suggesting LRD. These autocorrelograms are constructed from the 78-year fitting period; had the complete 131-year record been used, the peaks of autocorrelograms would be higher, indicating enhancement of both the periodicity and LRD. LRD is better seen in Fig. 4, which depicts higher lag autocorrelations of monthly flows for lags that are multiples of 12 (corr[X_i, X_{i+12j}]), so as to eliminate periodicity, as well as autocorrelations of annual flows (corr[Y_i, Y_{i+j}], where $Y_i := X_{12(i-1)+1} + \ldots + X_{12i}$, the annual flow at year i). The presence of LRD implies much higher uncertainty than in classical statistics, as well as bias in classical statistical estimators (Koutsoyiannis, 2003). Therefore, in Fig. 4, two series of estimates of ρ_j have been plotted, the classical statistical ones (marked as "empirical classical") and the adapted ones (marked as "empirical SSS" where SSS stands here for simple scaling statistics; see below) that recover from bias (Koutsoyiannis, 2003). In addition to empirical estimates, some model curves are also plotted, which will be discussed later. All panels

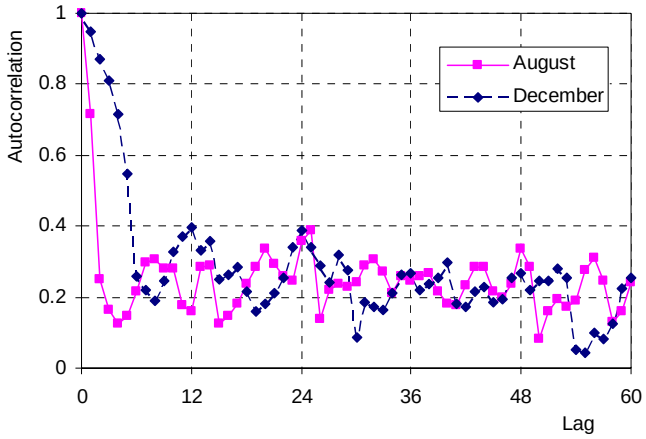

Fig. 3 Correlation coefficients of the transformed (by (9)) monthly flows, i.e. corr[Z_i, Z_{i+j}] for i = 1 (August) and 5 (December) and lag j up to 60.

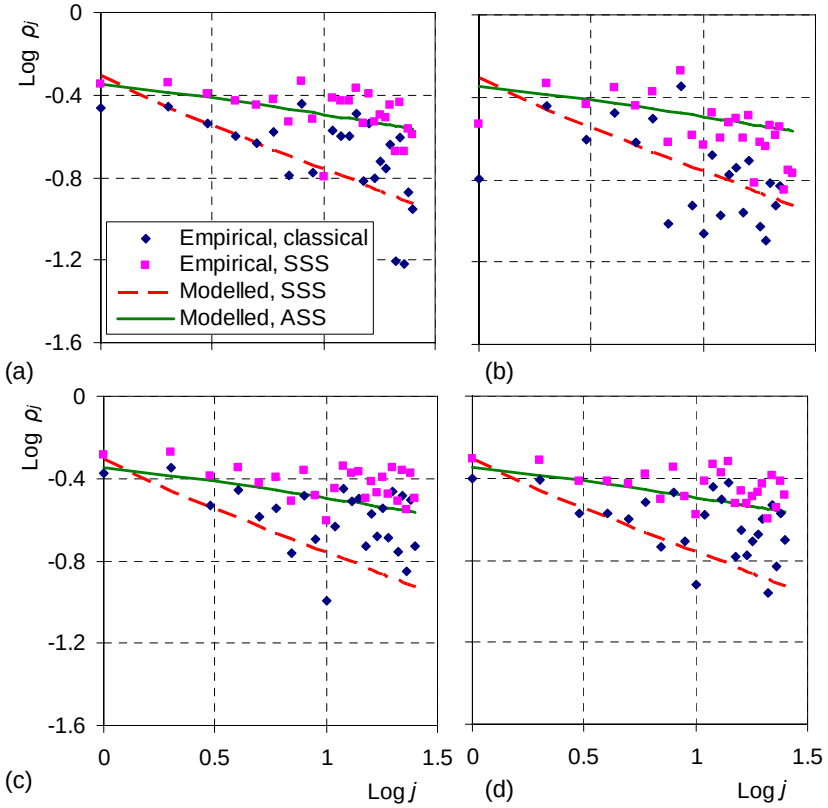

Fig. 4 Logarithmic plots of autocorrelations of annual flows (corr[Y_i, Y_{i+j}]) and monthly flows (corr[X_i, X_{i+12j}]) *vs* j: (a) annual; (b) August; (c) December, untransformed time series; and (d) December, transformed time series.

in Fig. 4 verify the presence of LRD and interestingly, indicate that this behaviour is virtually the same in all months as well as annually.

All above observations support a modelling approach of a mixed type, with a cyclostationary description of dependence at small lags (different for each month) and a stationary description for large lags (same for all months). The former can be done easily, using a small number of empirical autocorrelation coefficients (as those shown in Tables 2 and 3 estimated from the fitting period). To specify this number, we use the notion of explained variance. From a detailed investigation, we

observed that the portion of total variance explained by two autocorrelations (ρ_1 and ρ_2) is increased considerably from the case of using only one (ρ_1) whereas the introduction of additional autocorrelation coefficients (ρ_3, ρ_4, …) essentially does not increase the explained variance. Thus, the values of ρ_1 and ρ_2 given in Table 3 suffice to describe the dependence for small lags. This should not be confused with the adoption of a periodic AR(2) or ARMA(1,1) model as happens typically in stochastic modelling using two autocorrelations. These models would imply SRD, while here our aim is to preserve LRD.

A simple stationary structure with LRD is the simple scaling stochastic process (SSS process) with autocorrelation (e.g. Koutsoyiannis, 2002):

$$\rho_j = (1/2)\,(|j+1|^{2H} + |j-1|^{2H}) - |j|^{2H} \approx H(2H-1)\,|j|^{2H-2} \tag{10}$$

where H is the so-called Hurst coefficient with values in the interval (0.5, 1) for positively autocorrelated processes. Here j is meant as the lag for the annual scale or the lag divided by 12 for the monthly scale. However, Fig. 4 suggests that autocorrelation in the Nile decays with lower rates than implied by equation (10). Therefore, we need to investigate it further, again using the ME principle.

To determine the dependence structure of a stochastic process, Koutsoyiannis (2005b) maximized average entropy on a range of timescales with appropriate constraints. That ME framework was revisited and advanced in light of the statistical behaviours observed in the Nile (Koutsoyiannis & Georgakakos, 2006). The results of the latter work are also used here. To summarize them, the entropy maximization is done on a time scale tending either to zero (a local setting) or to infinity (a global setting). Maximization of entropy is done numerically using a parametric form of the autocorrelation function, initially formulated in continuous time, as shown in Appendix 1. This parametric autocorrelation includes three components: a white noise term, an SRD term and an LRD term. This parametric representation is deliberately rich (it includes six parameters) in order to provide appropriate degrees of freedom for the entropy maximization.

Entropy maximization either at the local or the global setting results in virtually the same solution if two autocorrelation constraints are used. This solution is more complicated than SSS, but tends to SSS as scale increases. For this reason, we call it an asymptotic scaling stochastic process (ASS process). Both SSS and ASS structures are plotted in terms of the resulting autocorrelation functions in Fig. 4, also in comparison with empirical autocorrelations. Practically, the autocorrelation of either SSS and ASS is a power function of the lag; the difference is that in SSS both the slope and the intercept are dependent on each other (they are functions of the single parameter H, namely $2H-2$ and $\rho_1 = 2^{2H-1} - 1$, respectively) whereas in ASS the intercept ρ_1 does not determine the slope of the autocorrelation decay. Here the SSS and ASS models were determined for the annual series and also applied in the monthly series. Thus, the model curves in all panels of Fig. 4 are exactly the same. It seems that ASS in all cases is in better agreement with the empirical points and therefore we use this in the development of the model. Even though, due to the adopted rich parameterization, the ASS autocorrelation is nominally dependent on six parameters, in fact, after entropy maximization, two parameters suffice to describe it (i.e. the intercept and slope in the plots of Fig. 4, which correspond to the two autocorrelations that were used as constraints).

The link of short-term dependence, described by the two small lag autocorrelations, with the long-term dependence, described with the ASS model, will be done in Section 4.3 to construct an operational stochastic model for prediction as well as for simulation.

4.3 Model formulation

In accordance with (1) and the linearity assumption justified in Section 3, our stochastic model will be a general linear one, i.e.:

$$W = \boldsymbol{a}^T \boldsymbol{Z} + V \tag{11}$$

where $\boldsymbol{a}$ is a vector of weights and V is a random term assumed independent of $\boldsymbol{Z}$. The variable to

be predicted, W, and all items of the vector of explanatory variables, $\boldsymbol{Z}$, are normalized monthly flows. Specifically, assuming that the current time is $i-1$, denoting Z_i the normalized flow at time i and $Z_{(i)}$ the ith item of $\boldsymbol{Z}$, we will have:

$$W \equiv Z_i\,,\ Z_{(1)} \equiv Z_{i-1},\ Z_{(2)} \equiv Z_{i-2},\ Z_{(3)} \equiv Z_{i-12},\ Z_{(4)} \equiv Z_{i-24},\ \ldots,\ Z_{(m)} \equiv Z_{i-12(m-2)} \tag{12}$$

Thus, the first two items of $\boldsymbol{Z}$ are the nearest in time normalized monthly past flows, whereas all other items are normalized past flows of the same month of the year as the month in time i. With this composition of $\boldsymbol{Z}$, the model takes account of both long-term and short-term dependence. To account for LRD as much as possible, we should make m as large as possible. In fact, the size m of $\boldsymbol{Z}$ is determined by the available data record that conditions prediction. In our case, $m - 2 = 78$ (the length of the fitting period), so $m = 80$. It should be emphasized that the size m is not at all related to the number of parameters to be estimated from the data, and there is no reason to seek parsimony in this case.

To specify the model we need to determine the weights $\boldsymbol{a}$ and the statistical characteristics of V. Assuming (for convenience) that W and $\boldsymbol{Z}$ have zero mean and unit variance, V will have zero mean, variance var[V] < 1, and normal distribution. Multiplying both sides of (11) by $\boldsymbol{Z}^T$ and taking expected values we obtain:

$$\boldsymbol{a}^T = \boldsymbol{\eta}^T \boldsymbol{h}^{-1} \tag{13}$$

where $\boldsymbol{\eta}$:= cov[W, $\boldsymbol{Z}$] and $\boldsymbol{h}$:= cov[$\boldsymbol{Z}$, $\boldsymbol{Z}$]. Squaring both sides of equation (11) and taking expected values we obtain:

$$\text{var}[V] = 1 - \boldsymbol{\eta}^T \boldsymbol{h}^{-1} \boldsymbol{\eta} = 1 - \boldsymbol{a}^T \boldsymbol{\eta} \tag{14}$$

We observe that the vector $\boldsymbol{\eta}$ contains 80 monthly autocorrelation items, the lag one and two, which are model parameters estimated from the data, and the lag 12, 24, etc., which, as described above, are equal to the lag 1, 2, … autocorrelations of the annual flows. The latter are determined from the ASS model described in Section 4.2 in terms of a couple of free parameters (annual autocorrelations for two lags). Thus the first two items of $\boldsymbol{\eta}$ change from month to month whereas all others are the same for all months. The matrix $\boldsymbol{h}$ contains numerous items ($80 \times 80 = 6400$ for each month). However, most of them (the lower 78×78 part of the matrix) are determined from the ASS model and a few more (the upper 2×2 part of the matrix) contain lag one and two autocorrelations already appearing in $\boldsymbol{\eta}$. The remaining part of the matrix (two, 2×78 areas, symmetric to each other because $\boldsymbol{h}$ is symmetric) contains unknown autocorrelations (cov[W, Z_i] for several Z_i).

According to prevailing practices in stochastic modelling, these unknown autocorrelations, whose number is very large ($12 \times 2 \times 78 = 1872$), would be estimated from the data. Even though this is technically feasible (and done in some cases such as in most disaggregation models) it makes no sense, given that the available data values are $12 \times 78 = 936$, i.e. half the number of these unknown autocorrelations.

Here we propose that these parameters should be left "unestimated" in the statistical sense and should be calculated by applying the ME principle. In this way, no additional parameter is introduced in the stochastic model. As shown in Appendix 2, the entropy maximization in this case has an easy analytical solution that can be formulated as a generalized Cholesky decomposition of the matrix $\boldsymbol{h}$ (assuming that $\boldsymbol{h} = \boldsymbol{b}\,\boldsymbol{b}^T$, where $\boldsymbol{b}$ is a lower triangular matrix to be calculated). In this case the total number of autocorrelation parameters to be estimated statistically does not exceed $(12 + 1) \times 2 = 26$ (two autocorrelations per month plus two parameters for the annual autocorrelation function) for the entire model; thus, the proposed model is indeed parametrically parsimonious.

After the calculation of the matrix $\boldsymbol{h}$, all other computational effort is trivial (typical matrix operations). Model (11) can perform either in forecast mode or in simulation mode. To apply model (11) to obtain a point prediction W for the observed values $\boldsymbol{Z} = \boldsymbol{z}$, it suffices to set $V = 0$; the resulting value of W from (11) will be the expected value conditional on $\boldsymbol{Z} = \boldsymbol{z}$. Interval predictions can be easily derived analytically based on the distribution of V which is normal with zero mean and variance var[V]. Furthermore, stochastic simulation is also easily performed using the same model with V generated from the normal distribution.

4.4 Results

Figure 5 provides a graphical depiction of the vector of weights $\boldsymbol{a}$ for two months. Due to the higher value of lag one autocorrelation in December (0.94 against 0.71 in August) the relative weight of distant lagged past flows is much smaller in December than in August. The entire picture of weights changes significantly from month to month (cyclostationarity) even though most parts of matrices $\boldsymbol{\eta}$ and $\boldsymbol{h}$ represent stationary components. Generally, the weights decrease with the increase of lag, but this is reversed for very high lags. This seems counter-intuitive but it is totally justified: the non availability of information for lags higher than 78 years results in relatively higher weights near the 78-year lags.

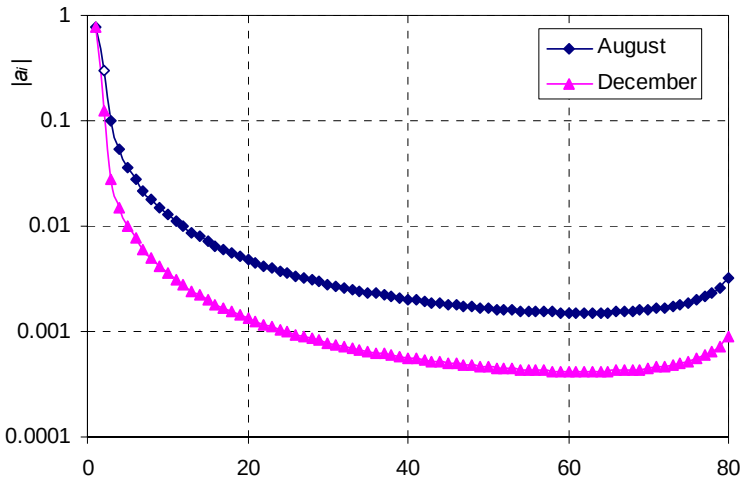

Fig. 5 Graphical depiction of the vector of weights *a* (coefficients of time delayed standardized flows in the multivariate linear stochastic model) for the indicated months; full points indicate positive values and the single empty point indicates a negative value.

Based on the 12 vectors of weights, the model was applied to predict the flows of the 53-year validation period. Each time, the most recent 78-year historical information was used to condition the prediction. The results are shown graphically in Fig. 6, both in terms of natural and standardized (by month) values, and indicate a very satisfactory proximity with actual values. A numerical index measuring performance is the attained coefficient of efficiency:

$$C_E = 1 - E[(W - X)^2]/\text{var}[X] \quad (15)$$

where X is the actual variable that is predicted by W. As a benchmark model for assessing prediction skills of all models we use the so-called zero order (ZO) prediction, in which the flow of the current month is taken as the forecast for the next month. As shown in Table 4 (second column), the performance index of the stochastic model in its above described configuration (S1) is very high, impressively higher than in the ZO prediction (also shown in Table 4).

In addition to this full configuration of the stochastic model, two additional ones were examined (see Table 1), whose performances are also shown in Table 4. In configuration S2, which is similar to S1 but without normalizing transformation, the performance is only slightly lower than in S1. Even performance of the stochastic model S3, which is a typical PAR(2) process without a normalizing transformation, is good, but inferior to those of S1 and S2.

5 DETERMINISTIC MODELS

5.1 Analogue model

As described in Section 3, the logic and the algorithm of the analogue model are very easy (Georgakakos & Yao, 1995; Kantz & Schreiber, 1997; Yao & Georgakakos, 2001). In operational

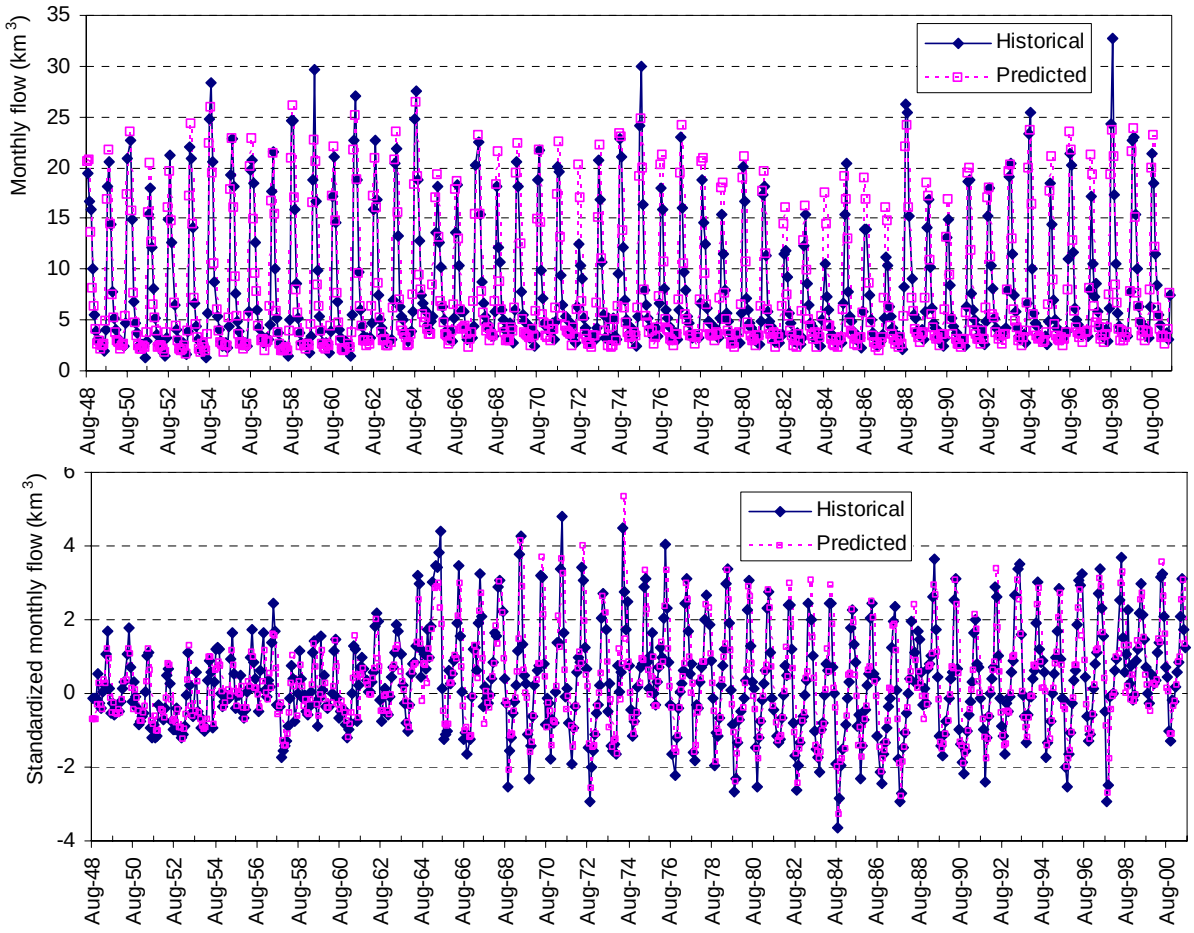

Fig. 6 Graphical depiction of the proximity of monthly predictions of model S1 to historical values: (a) natural values; and (b) monthly standardized values.

Table 4 Coefficients of efficiency of the different prediction models for the validation period (53 years).

Model	Untransformed values	Logarithmically transformed values	Monthly standardized untransformed values
ZO	0.327	0.463	–0.237
S1	0.911	0.904	0.673
S2	0.907	0.899	0.675
S3	0.884	0.884	0.624
A1	0.840	0.613	–0.145
A2	0.847	0.623	–0.126
A3	0.879	0.851	0.490
C1	0.888	0.878	0.583
C2	0.775	0.791	0.280
C3	0.859	0.849	0.472

mode, the only difference from the general description in Section 3 is that a number of neighbours $z^1, \ldots, z^n$, instead of a single vector z^1, is used and the prediction w is extracted as the average of $w^1, \ldots, w^n$, the states next (in time) to the latest coordinate of $z^1, \ldots, z^n$, respectively. The number n can be either fixed or varying determined in a manner that the vectors z^i have distance from z smaller than a threshold; here the first option has been used. The vector z is formed from the current state and some earlier ones whereas the vectors z^i are sought in the calibration period exclusively.

Thus, the analogue model involves no parameters except the size of the vector z (embedding dimension, m) and the number of neighbour's n. These adjustable quantities are determined by a trial-and-error procedure aiming at finding the optimal m and n that make the prediction error minimum for the verification period. Application of the method with the Nile flows resulted in the variation of the coefficient of efficiency in the verification period with m and n that is shown in Fig. 7. It is generally observed that if we exclude too low values (i.e. m = 1 and n = 1–2) the efficiency is very good. Two local optima were found corresponding to (m = 12, n = 11) and (m = 13, n = 24). The efficiencies of these two model configurations, abbreviated as A1 and A2, respectively (see Table 1) are shown in Table 5.

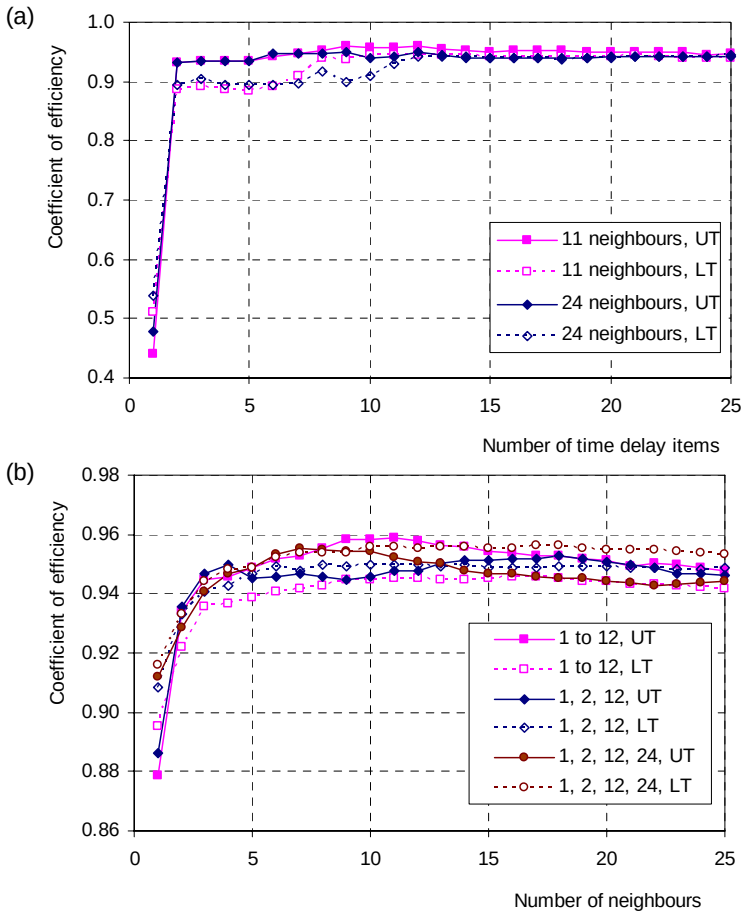

Fig. 7 Coefficient of efficiency attained by the analogue model for the verification period, as a function of (a) number of delay items assuming fixed (11 or 24) number of neighbours, and (b) the number of neighbours assuming the indicated delay items; UT untransformed (natural) values; LT logarithmically transformed values.

Table 5 Coefficient of efficiency of the optimal configurations of the analogue model in the verification period; all coefficients refer to natural (not standardized) series.

Model	Untransformed values	Logarithmically transformed values
A1	0.959	0.945
A2	0.945	0.942
A3	0.955	0.954

We also studied an additional model configuration, inspired by the stochastic model. In this, to construct the vector z we assumed (similar to (12)) a variable (rather than constant) time delay,

i.e. $\boldsymbol{z} = [z_{i-1}, z_{i-2}, z_{i-12}, z_{i-24}, \ldots, z_{i-12(m-2)}]^T$. In this case, however, we cannot use a high m (like 78 in the stochastic model) because we would run out of a pool of neighbours. Thus, we only tested the cases m = 3 and 4; with these values we cannot anticipate to capture the long-term dependence properties of the process, but only to simplify the model (using four instead of 24 terms for the same effective total lag). The resulting efficiencies for these two cases are also plotted in Fig. 7 and are comparable to that of the constant time lag cases. Among the several configurations of this type shown in Fig. 7, the optimal was that with m = 4 and n = 7; this has also been included in Table 1 and Table 5 abbreviated as A3. The performances of all model configurations for the validation period are shown in Table 4.

5.2 **Connectionist model**

The connectionist model used in this study follows the logic described in Section 3; its details are described in Georgakakos & Yao (1995). In our case study, structures with one or two hidden layers have been examined. The model fitting, metaphorically known as "training" or "learning", is a nonlinear optimization procedure that minimizes fitting errors. In this case it was executed by the "error backpropagation" method which is a version of a gradient descent method.

As opposed to the analogue model case, in which the natural flows were used, here the flows were standardized by the mean and standard deviation of each month. To avoid over-fitting (i.e. the use of too many components of elementary functions, a common propensity of connectionist models) an early stopping method was used combined with two fitting measures: the calibration error (in the calibration period) and the verification error (in the verification period; Georgakakos & Yao, 1995). Typically, the calibration error decreases steadily while the verification error initially decreases and eventually increases, exhibiting a minimum. Model calibration is typically terminated when the verification error achieves a minimum value.

Several model configurations were tested, which make two groups. In the first group a constant time delay (1 month) was assumed, the number of inputs varied from 1 to 15, the hidden layers from 1 to 2, and the hidden nodes in each layer from 1 to 15. In the second group the time delay was variable (as in the analogue model), the number of inputs was fixed to 4, the hidden layers were 1 or 2, and the hidden nodes in each layer varied from 1 to 10. The tradeoff of the two fitting measures for all examined configurations and the Pareto front formed are shown in Fig. 8.

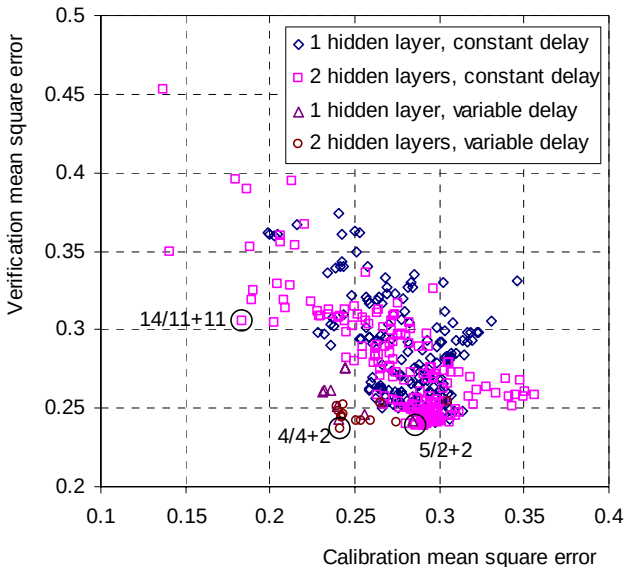

Fig. 8 Plot of the attained verification error *vs* the attained calibration error of a series configurations of the connectionist model with 1–15 input nodes, 1 or 2 hidden layers, and 1–15 hidden nodes in each layer. The circled points in the Pareto front, for which the number of input and hidden nodes are marked, depict the solutions further explored (from left to right: C2, C3, C1).

Table 6 Performance indices of the optimal configurations of the connectionist model for the calibration and verification periods; all coefficients refer to standardized series.

Model	Mean square error:		Coefficient of efficiency:	
	Calibration	Verification	Calibration	Verification
C1	0.289	0.241	0.749	0.435
C2	0.183	0.309	0.842	0.277
C3	0.241	0.240	0.794	0.438

From the solutions lying in the Pareto front, three were chosen as model configurations C1, C2 and C3, whose characteristics are shown in Fig. 8 and Table 1. Further, in Table 6, which depicts several fitting criteria, we observe that the performance of all configurations is very good (having in mind that the values given in the table are for the standardized variables) with C3 slightly outperforming the other two. The performances of all model configurations for the validation period are shown in Table 4.

6 MODEL INTERCOMPARISON

The intercomparison of models in terms of their prediction skill is made for the 53-year validation period, which was not used in any fitting procedure in any model. Three performance indices: the coefficients of efficiency of untransformed values, logarithmically transformed values, and monthly standardized untransformed values, are shown in Table 4. By all indices, all models in all their configurations give predictions much better than the ZO prediction. Among them, the stochastic models S1 and S2 have the best performance and are followed by the connectionist model C1 (which has almost equal performance with S3) and the analogue model A3.

In terms of simplicity and ease of application, the analogue model is best. A spreadsheet environment suffices to develop, calibrate, and run it, and its development can take place very quickly. In particularly, configuration A3, which gives the best performance in the validation period among the three analogue models, is also the simplest (as it involves only four variables) and fastest in calibration and running. The next model group in terms of simplicity is the stochastic. Configuration S3 is very simple but even the full proposed model S1 is simple enough to be implemented in a spreadsheet environment. Here we deliberately discussed the model in depth without simplifications. However, in a practical application simplifications are possible. For instance, the SSS model could be adopted by default, without entropy maximization. Another option is to use a generalized parametric autocorrelation as in Koutsoyiannis (2000) and determine its parameters by fitting it (e.g. by least squares) to the empirical autocorrelogram, again without entropy maximization. In contrast, the connectionist approach is not simple and cannot be implemented on a spreadsheet.

In terms of model ability to perform in simulation mode, in addition to forecast mode, only the stochastic model provides this option; its procedure was discussed in Section 4.3. To illustrate this, a synthetic record of length equal to that of the historical was generated by model S1. Comparisons of the statistics of the synthetic and historical monthly records are given in Fig. 9, which indicates a satisfactory performance. Some discrepancies of the skewness and lag 12 autocorrelations during low flows are usual due to the small sample size; to match such statistics a longer sample size by several orders of magnitude is needed, particularly because of the increased sampling variability due to LRD.

The analogue model cannot operate in simulation mode because it soon converges to an "attracting" periodic trajectory, the same for all years. The connectionist model, when the number of nodes is small, behaves similarly to the analogue model resulting in an "attracting" periodic trajectory. For more than 15–20 hidden nodes, it produces irregular trajectories, which do not resemble, and are statistically dissimilar to, the historical flows.

In terms of potential insights into the process, we can argue that the stochastic approach offers some, especially when combined with the ME principle. The latter, as discussed above provides an

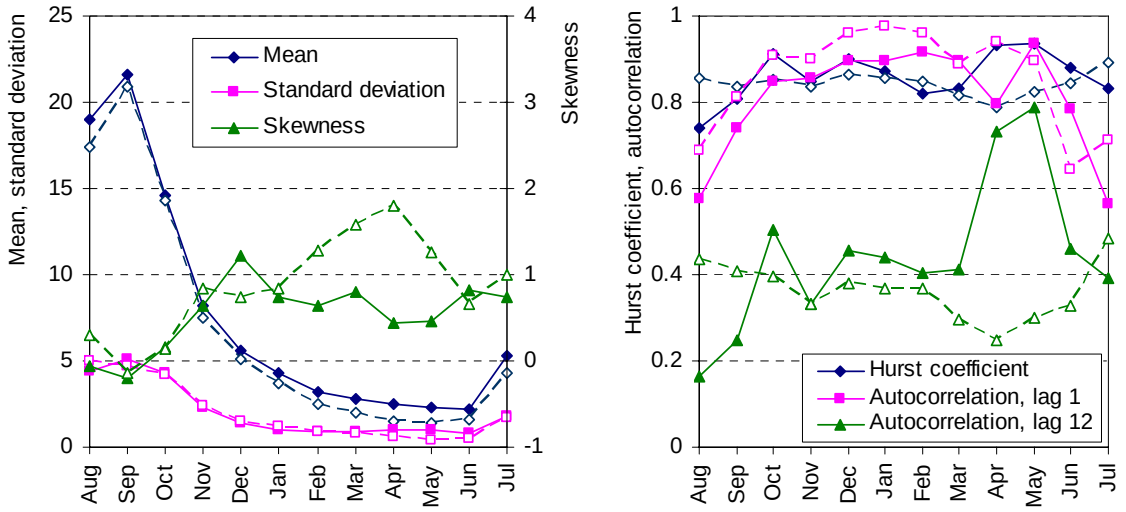

Fig. 9 Comparison of statistics of the historic monthly record (continuous lines) and a synthetic record of equal length generated by model S1 (dashed lines).

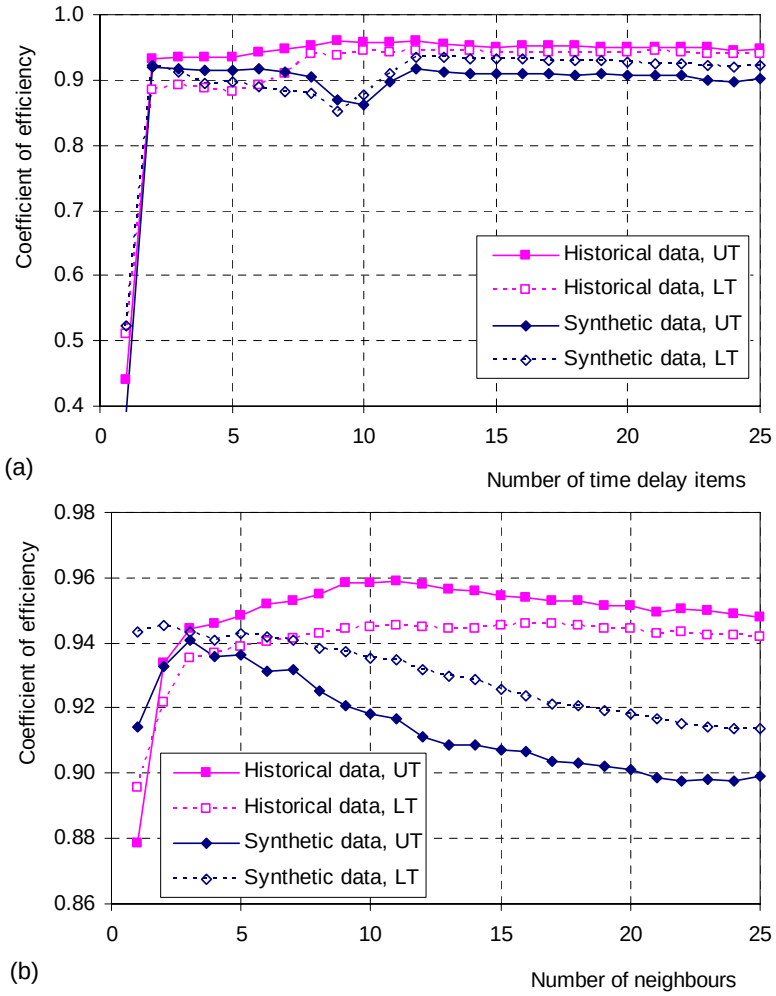

Fig. 10 Coefficient of efficiency attained by the analogue model for the verification period for synthetic data generated by model S1 in comparison to the respective values for the historic data, as a function of (a) number of delay items assuming fixed (11) number of neighbours, and (b) the number of neighbours assuming fixed (12) delay items; UT untransformed (natural) values; LT logarithmically transformed values.

explanation of the observed linearity on a stochastic setting, of the marginal distribution and particularly its tail, and of the dependence structure, particularly the long-term one. The parametric setting of the stochastic approach, along with these insights, offers the ability of controlling the entire procedure. In contrast, the deterministic approaches, which are data driven, do not offer any control and are black-box rather than insightful. Furthermore, the deterministic models cannot have any control on the distribution tails and cannot describe LRD for technical reasons related to their data-driven aspect: as described above, an attempt to construct time delay vectors going far to the past will result in lack of a pool of such vectors.

It has been argued that such or similar deterministic approaches (e.g. time delay embedding, see Koutsoyiannis, 2006) offer insights because they reconstruct the dynamics of the process based on the observed time series and uncover its deterministic attractor. This is true for simple low dimensional experimentation systems (e.g. with one positive Lyapunov exponent) but it is unlikely to be the case for complex natural processes, such as the flow of the Nile. The fact that such deterministic models can cast good predictions should not necessarily be given the interpretation that the process is governed by purely deterministic dynamics. To illustrate this we used the aforementioned synthetic flows generated by the stochastic model S1, to which we applied the analogue model. As shown in Fig. 10, despite the *a priori* known stochastic character of the inflows, the analogue model gave good predictions and its performance is comparable to that with the historical data. Thus, a good prediction does not necessarily imply deterministic dynamics.

7 CONCLUSION AND DISCUSSION

A general conclusion of this work is that it is always worth constructing a good stochastic model of a hydrological process. Such a model can operate in simulation mode as well in forecast mode and thus can support strategic planning as well as real time management of a hydrosystem. In addition, the development of a good stochastic model is closely linked with understanding of natural behaviours in a bidirectional manner: a good model presupposes understanding and also supports understanding by providing insights into natural behaviours. These behaviours include extreme phenomena (distribution tails) and temporal dependences, particularly LRD. It becomes obvious then that good modelling practices should depart from the typical ARMA formalism.

The principle of maximum entropy can largely support the development of stochastic models, providing both a logical foundation and computational tools. Here the principle was used four times, i.e. (a) to infer the marginal distribution of the process, (b) to explain and model the long-term dependence of the process, (c) to justify the (macroscopic) linearity in lagged flows of the Nile, and (d) to determine unknown covariances in the stochastic model structure. This study offers advances in (a) and (b) whereas to our knowledge propositions (c) and (d) are original. Particularly, proposition (d) offers a powerful yet very simple (because of the closed analytical solution) computational tool for the construction of a generalized stochastic model.

Deterministic modelling alternatives, which recently have been given great attention, are good practical tools, too. The analogue model in particular is very attractive due to its simplicity, non-parametric character and easiness to construct and apply. However, care is needed in interpretation. Good predictions by deterministic models do not necessarily mean consistency of the natural process with determinism. In this case study, all configurations of deterministic models gave performance inferior to the advanced stochastic model. Perhaps however, in another application or with the use of more advanced deterministic techniques, they may perform better. Still, however, deterministic models are inferior on other grounds, such as in supporting Monte Carlo simulation of hydrosystems or in deriving interval predictions, in describing and exploiting the long-term persistence, and in offering insights into the process.

It may seem counterintuitive that the particular stochastic model developed in this study, which largely relies on the principle of maximum entropy, in other words on the postulation of large uncertainty in nature, yields better forecasts than the deterministic models negating uncertainty. Perhaps stochasticity and the notion of maximum entropy explain natural behaviour better than determinism.

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APPENDIX 1

Entropy maximization to determine LRD

We assume a continuous time representation of the process of interest with the following three-component parametric autocovariance function:

$$\gamma(\tau) := \mathrm{cov}[X(t), X(t+\tau)] = \lambda_0\,\delta(\tau) + \lambda_1 \exp(-\kappa_1|\tau|) + \lambda_2\,(1 + \kappa_2\,\beta\,|\tau|)^{-1/\beta} \tag{A1}$$

where $\delta(\tau)$ is the Dirac delta function and λ_0, λ_1, λ_2, κ_1, κ_2 and β are parameters. The three components of $\gamma(\tau)$ are respectively, a white noise term (Dirac delta), a SRD term (Markovian or exponential function of time lag) and an LRD term (generalized power function of time lag with exponent β manifesting LRD unless $\beta = 0$). For $\beta = 0$ the third term becomes a second exponential term and represents SRD at a second characteristic time scale. For $\beta > 1$ this term approaches the SSS autocovariance with Hurst exponent $H = 1 - 1/(2\beta)$. When $\lambda_0 = \lambda_1 = 0$, $\lambda_2 \neq 0$, $\beta > 1$ and $\kappa \to \infty$, the process becomes precisely SSS.

Let X_i denote the process at discrete time with scale d, i.e.

$$X_i := \frac{1}{d}\int_{(i-1)d}^{id} X(t)\,\mathrm{d}t \tag{A2}$$

Then, the following equations can be obtained from typical calculus of stochastic processes:

$$\gamma_0 := \mathrm{var}[X_i] = \lambda_0 / d + 2\,\varepsilon_1\,(\xi_{1,1} + \kappa_1 d - 1) + 2\,\varepsilon_2\,[\xi_{2,1} - (2\beta - 1)\,\kappa_2 d - 1] \tag{A3}$$

$$\gamma_1 := \mathrm{cov}[X_i, X_{i+j}] = \varepsilon_1\,(\xi_{1,j-1} + \xi_{1,j+1} - 2\,\xi_{1,j}) + \varepsilon_2\,(\xi_{2,j-1} + \xi_{2,j+1} - 2\,\xi_{2,j}), j > 0 \tag{A4}$$

where:

$$\varepsilon_1 := \lambda_1 / (\kappa_1^2\,d^2),\ \varepsilon_2 := \lambda_2 / [(\beta - 1)(2\beta - 1)\,\kappa_2^2\,d^2] \tag{A5}$$

$$\xi_{1,j} := \exp(-\kappa_1\,d\,j),\ \xi_{2,j}: = (1 + \kappa_2\,\beta\,d\,j)^{2\,-\,1/\beta} \tag{A6}$$

The details of the entropy maximization procedure, according to which the six parameters of the autocovariance function are determined, are given elsewhere (Koutsoyiannis & Georgakakos, 2006). A summary of the method is as follows: (a) the maximization of entropy is done numerically based on the above parametric representation of the dependence structure for a time scale tending to 0 or ∞; (b) numerically, the limiting timescales are approximated by $d = 2^{\pm s}$ where it was chosen that $s = 7$; (c) the constraints used in the maximization deal with known variance (the mean is not necessary if the distribution is normal), and known annual autocorrelation at lag one and at a greater lag; and (d) in addition to these equality constraints, an inequality constraint is used, i.e. that the information gain (see definition in Koutsoyiannis, 2005b) at time scale tending to zero is greater than at any positive time scale (meaning that predictability at any timescale should be lower than that instantaneously after the measurement).

Application of the procedure with the 78-year record of standardized annual flows with constraints var[Y] = 1, $\rho_1 = 0.45$ and $\rho_4 = 0.37$ (which are the SSS sample estimates and differ from the values in Table 2) resulted in $\lambda_0 = 0.00484$, $\lambda_1 = 24.87$; $\lambda_2 = 1$, $\kappa_1 = 46.80$, $\kappa_2 = 0.179$, and $\beta = 5.25$. This solution has the properties of an ASS process as discussed in the text.

APPENDIX 2

Entropy maximization to determine unknown covariances in the stochastic model structure

Let $\boldsymbol{Z}^+ = [W, \boldsymbol{Z}^T]^T$ be the vector that contains all variables of the problem, assumed to be standardized. Its covariance matrix is:

$$\boldsymbol{c} := \text{cov}\,[\boldsymbol{Z}^+, \boldsymbol{Z}^+] = \begin{bmatrix} 1 & \boldsymbol{\eta}^T \\ \boldsymbol{\eta} & \boldsymbol{h} \end{bmatrix} \tag{A7}$$

From a known result for Gaussian processes (Papoulis, 1991, p. 564), the joint entropy of $\boldsymbol{Z}^+$ is $\varphi^+ = \ln(\sqrt{(2\pi e)^{s+1}\,|\boldsymbol{c}|})$ where $|\boldsymbol{c}|$ is the determinant of $\boldsymbol{c}$. It is reasonable to assume that the unknown items of $\boldsymbol{h}$ are those that maximize φ^+, or equivalently $|\boldsymbol{c}|$. Any different consideration would imply that the information we have about $\boldsymbol{Z}^+$ is more than contained in the known elements of $\boldsymbol{c}$.

The maximization of φ^+ can be performed very easily. Since $\boldsymbol{c}$ is a symmetric matrix, it can be written as $\boldsymbol{c} = \boldsymbol{b}\,\boldsymbol{b}^T$ where $\boldsymbol{b}$ is a lower triangular matrix, known as the square root of $\boldsymbol{c}$. Then:

$$|\boldsymbol{c}| = \prod_{i=1}^{s+1} b_{ii}^2 \tag{A8}$$

so that maximization of $|\boldsymbol{c}|$ is equivalent to maximization of the product of the diagonal elements of $\boldsymbol{b}$. Recall that the elements of $\boldsymbol{b}$ are calculated from $\boldsymbol{c}$ with a step-by-step algorithm (Cholesky decomposition, e.g. Bras & Rodriguez-Iturbe, 1985, p. 96) for growing i and then j, so that:

$$b_{ij} = \frac{c_{ij} - \sum_{l=1}^{j-1} b_{il}\,b_{jl}}{b_{jj}} \text{ if } j < i;\quad b_{ii} = \sqrt{c_{ii} - \sum_{j=1}^{i-1} b_{ij}^2} \text{ if } j = i;\quad b_{ij} = 0 \text{ if } j > i \tag{A9}$$

Clearly then, maximization of the diagonal element b_{ii} demands that all "free" b_{ij} (which correspond to an unknown c_{ij}) should be zero. Thus, only a small modification of the algorithm is needed for the non-diagonal elements: if c_{ij} is unknown then $b_{ij} = 0$, else b_{ij} is calculated from (A9). Furthermore, one may observe that since Var[V] is related to the entropy of $\boldsymbol{Z}^+$ conditional on known $\boldsymbol{Z}$, it can be (alternatively to (14)) estimated from (Papoulis, 1991, pp. 500, 568):

$$\text{var}[V] = |\boldsymbol{c}|/|\boldsymbol{h}| \tag{A10}$$

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