European Geosciences Union General Assembly 2008
Vienna, Austria, 13-18 April 2008
Session IS40: Geophysical Extremes: scaling versus nonstationarity

Emergence of antipersistence and persistence from a deterministic toy model

D. Koutsoyiannis
Department of Water Resources and Environmental Engineering
National Technical University of Athens
(www.itia.ntua.gr/dk)

1. Abstract

A toy model is developed to demonstrate the emergence of antipersistence and persistence using simple deterministic dynamics. Because of its simplicity, it may be useful in understanding these behaviours and in avoiding misinterpretation of more complex natural systems. A hypothetical plain is assumed with water stored in the soil, which sustains some vegetation. Each year a constant amount of water enters the soil and the potential evapotranspiration is also constant, but the actual evapotranspiration varies following the variation of the vegetation cover, which in turn varies with soil water. The vegetation cover and the soil water storage are the two state variables of the system. The system dynamics is expressed by very simple equations. It is demonstrated that the system trajectory, as seen from synthesized time series, is characterized by antipersistence or fluctuations around the mean value with fast recovery of the mean. The fluctuations seem to be periodic but longer series reveal that there is no constant period. This behaviour reminds time series of phenomena that have been called "oscillations" such as the El Nino Southern Oscillation. On the other hand, the series of consecutive peaks of the system storage exhibits large and long excursions of local average from the overall mean, a behaviour known as long-term persistence or scaling behaviour. The produced trajectories give the impression of nonstationary time series but there is nothing nonstationary in the model, which involves only three parameters constant in time, i.e. the constant infiltration and potential evaporation rates, and a standardizing parameter for soil moisture.

2. The toy system and its dynamics

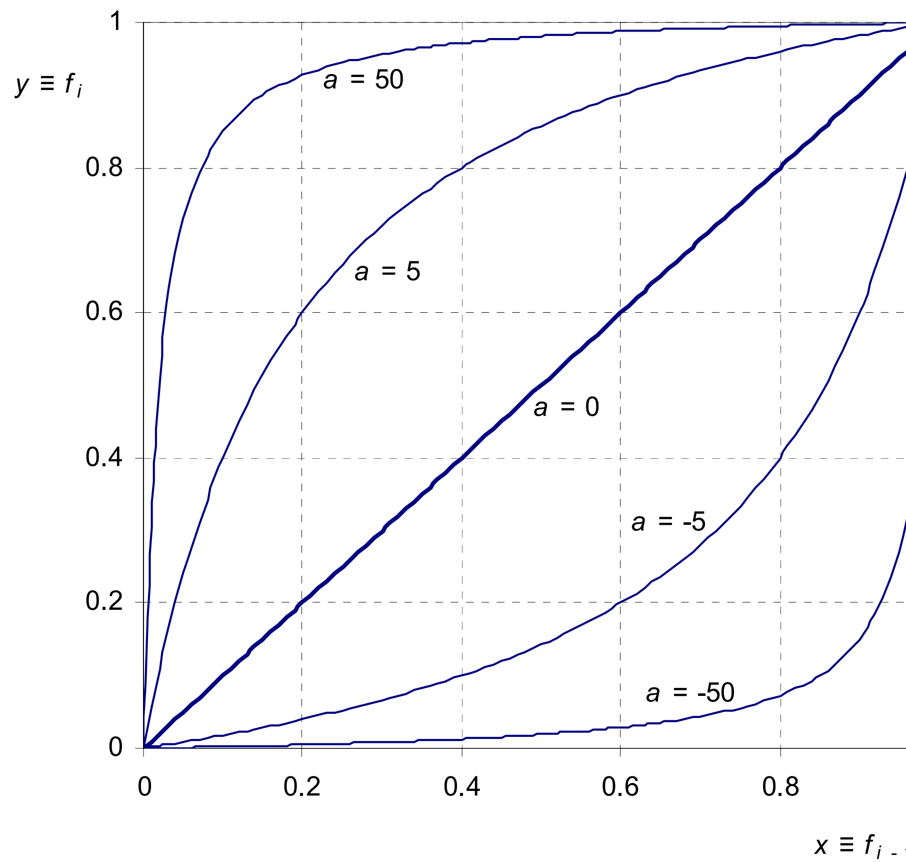
- The toy model refers to a fully deterministic system, deliberately made extremely simple.
- This system is a natural plain with water stored in the soil, which sustains some vegetation.
- Each year a constant amount of water $I = 250$ mm enters the soil and the potential evapotranspiration is also constant, PET = 1000 mm. The actual evapotranspiration is $E \leq \text{PET}$.
- A fraction f of the total plain area is covered by vegetation, and the evapotranspiration rate in this area equals PET and in all other area is zero (assuming no route of soil water to the surface), so that in the entire plain, the average actual evapotranspiration will be

  $$E = \text{PET}\, f$$
- The system is described by two state variables, which can vary in time: the vegetation cover f and the soil water s, for which we set $s = 0$ for a certain reference level, so that $s > 0$ stands for soil water excess and $s < 0$ for soil water deficit.
- If $i = 1, 2, \ldots$ denotes time in years, then the water balance equation is

  $$s_i = s_{i-1} + I - \text{PET}\, f_{i-1}$$
- If $f = I / \text{PET} = 0.25$ then $E = I = 250$ mm (input equals output) and the system stays at an equilibrium; the water stored in the soil stays at a constant value.
- If at some time, $f \neq 0.25$, then the state variables will vary in time.

3. System dynamics (2)

- We need one more equation to fully describe the system; naturally, this should be sought in the dynamics of grow and decay of plants, which however may be too complicated. Here we approach it in an extremely simplified, conceptual manner.
- We set a basic desideratum that f should increase when $s > 0$ (there is plenty of soil water and the vegetation will tend to expand) and decrease otherwise. A second desideratum is $0 \leq f \leq 1$.
- Such desiderata are fulfilled by the curves shown in the figure and described by the equation below it, which takes an input x and produces an output y, depending on a parameter a, positive or negative.
- If in this equation we substitute f_{i-1} for x, f_i for y, and some increasing function of s_{i-1} for a, then we obtain an equation that is conceptually consistent with our desiderata.
- For the latter (which should be 0 when $s_{i-1} = 0$) we set $a \equiv (s_{i-1}/s^*)^3$, where $s^* = 100$ mm is a standardizing constant.



$$y = g(x, a) := \frac{\max(1 + a, 1)x}{\max(1 - a, 1) + ax}$$

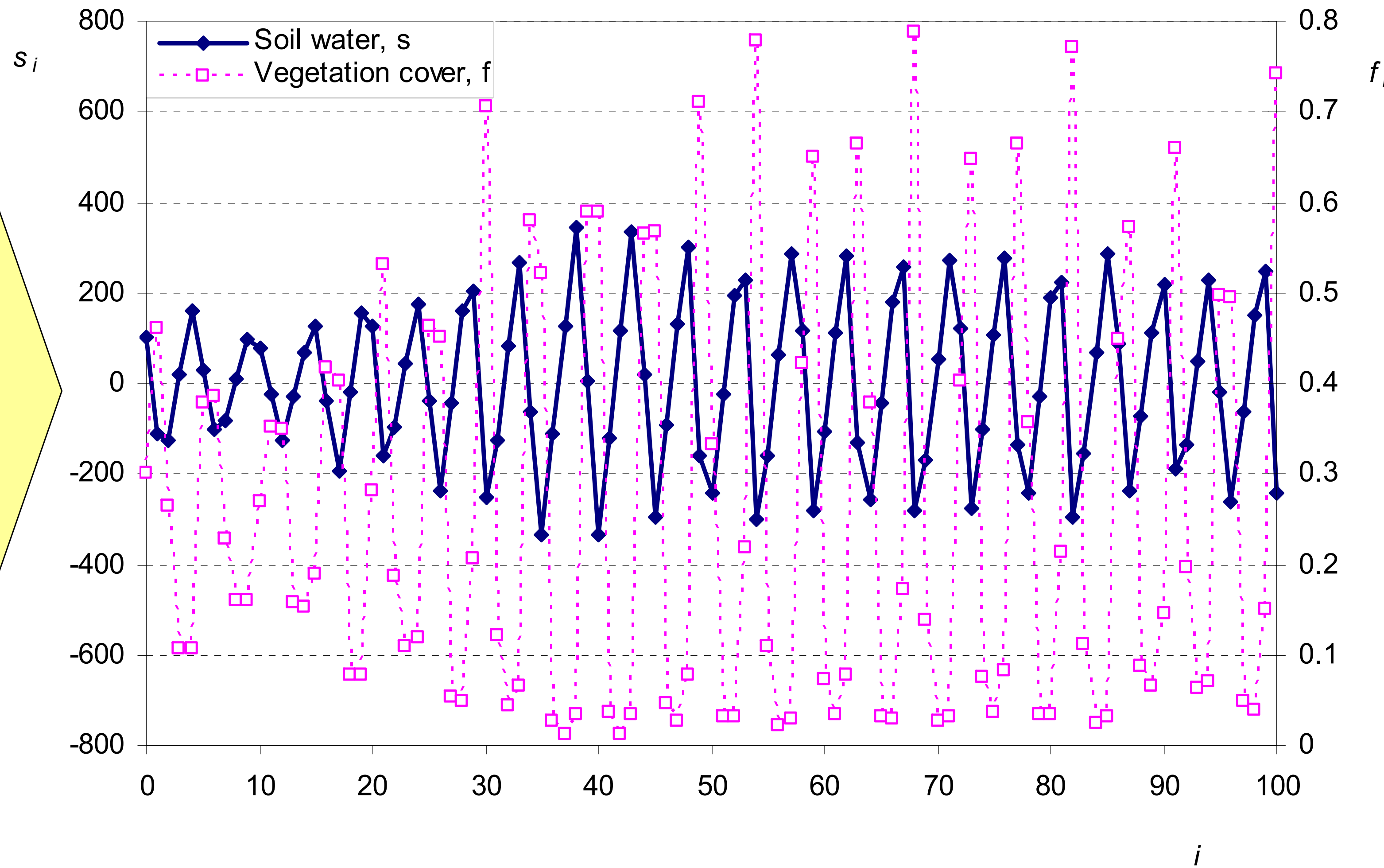
For $a \equiv (s_{i-1}/s^*)^3$, $x \equiv f_{i-1}$, $y \equiv f_i$:

$$f_i = \frac{\max(1 + (s_{i-1}/s^*)^3, 1)\, f_{i-1}}{\max(1 - (s_{i-1}/s^*)^3, 1) + (s_{i-1}/s^*)^3 f_{i-1}}$$

4. General behaviour

The system produces irregular trajectories, where the vegetation cover f fluctuates around 0.25 (the equilibrium point) and the soil water s fluctuates around 0. These fluctuations seem to have a period of roughly 4-5 years but are not perfectly periodic.

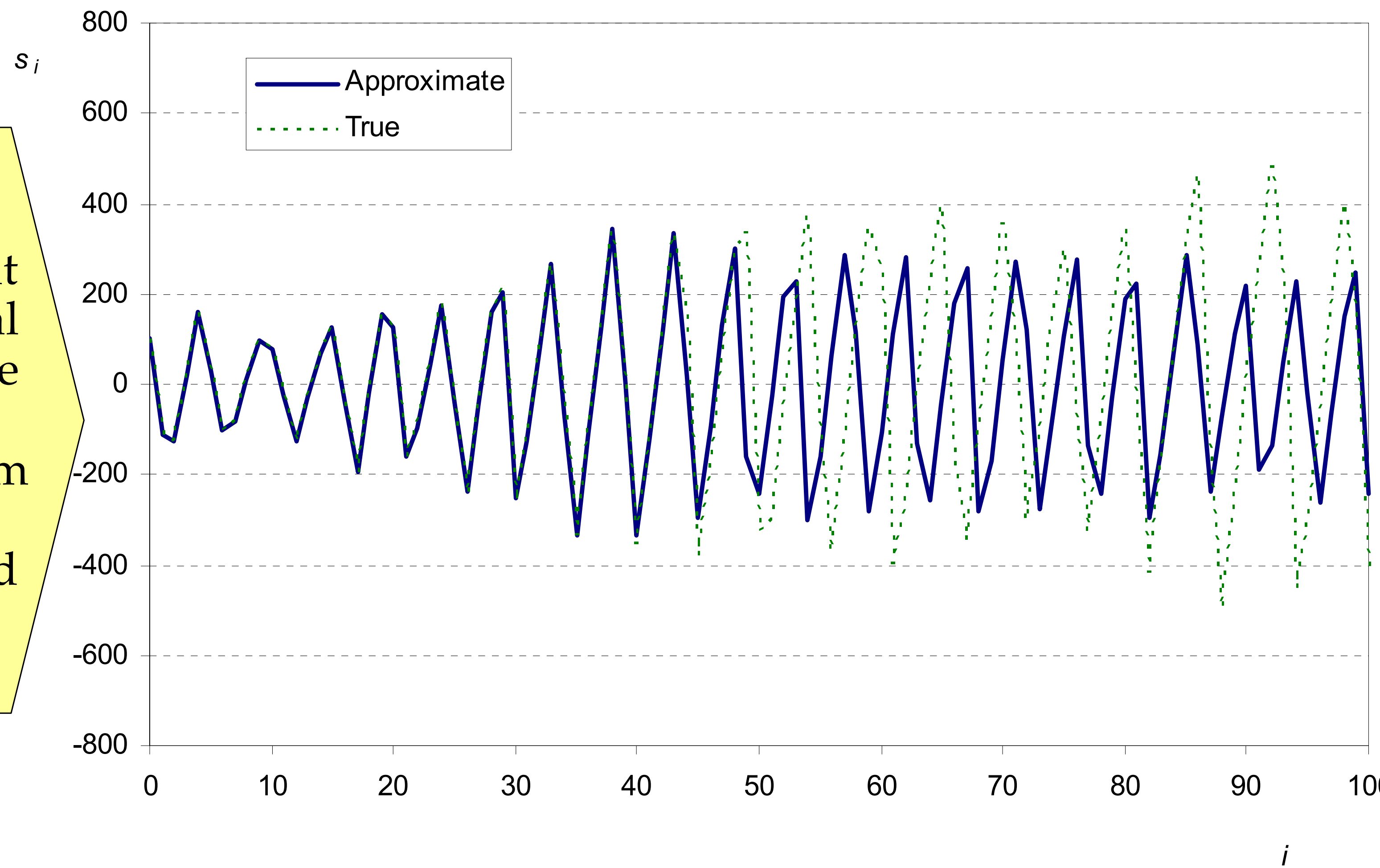
System evolution for time up to 100 for $f_0 = 0.30$ and $s_0 = 100$ mm



5. Predictability of the system state

The system is sensitive to initial conditions, i.e. chaotic: a small error of 0.01 mm in the initial storage is magnified to ~500 mm in 60 time steps; thus the model, despite being simple, cannot give good predictions for long time horizons.

Storage evolution assuming that the true initial conditions are $f_0 = 0.2999$, $s_0 = 100.01$ mm and are approximated as $f_0 = 0.30$, $s_0 = 100$ mm

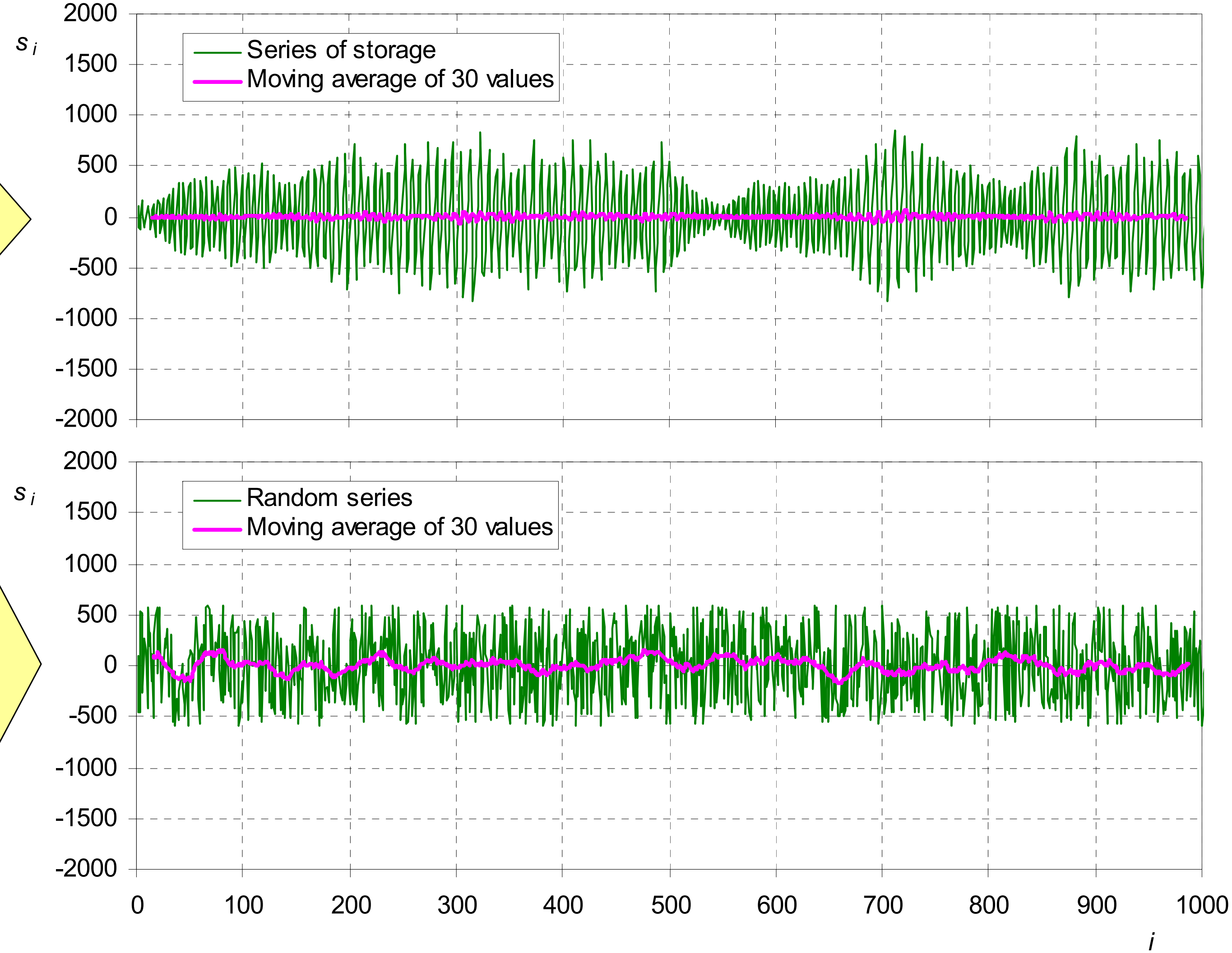


6. Long-term system behaviour

The system exhibits a strongly **antipersistent** behaviour: the plot of moving average is virtually a horizontal straight line (for comparison, in a purely random series it is a curly line)

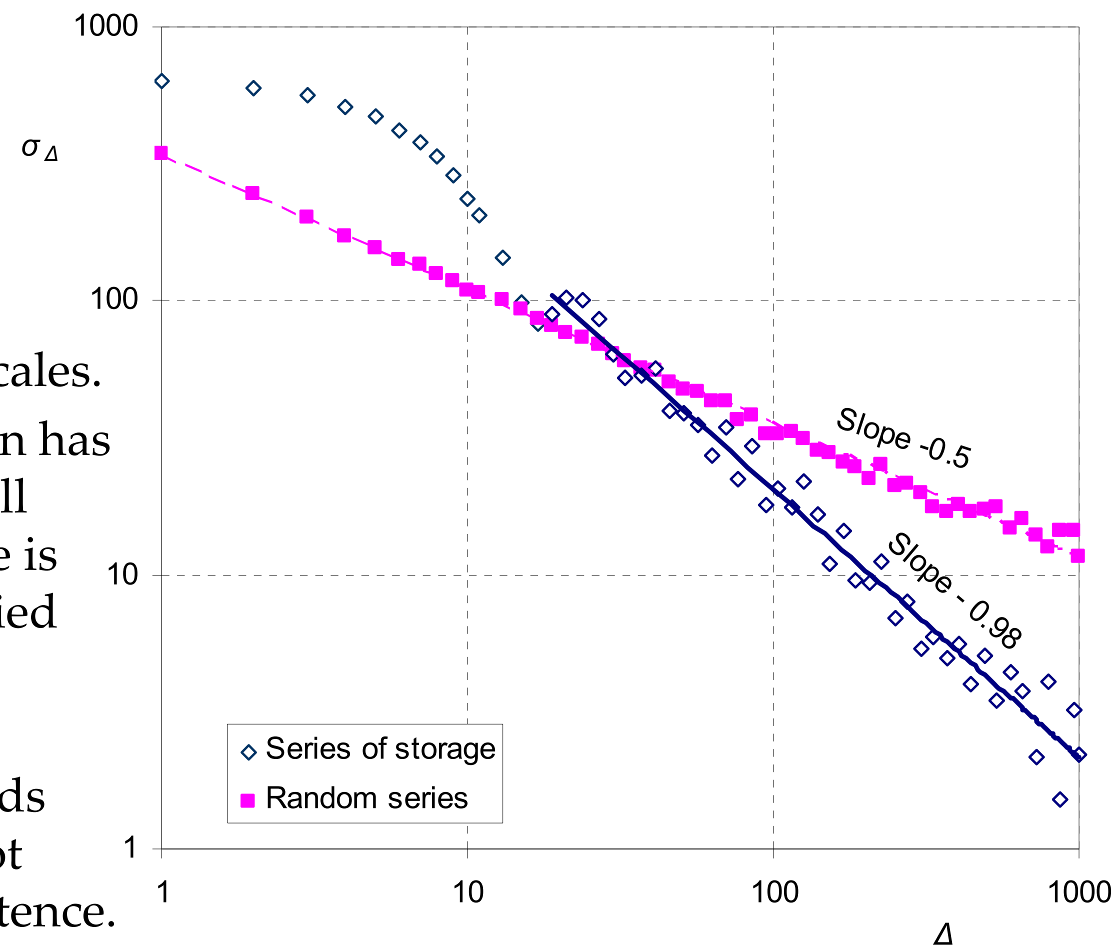
Evolution of the system storage s_i (in mm) of for time up to 1000

A series of random numbers in the interval (-800, 800) with mean and standard deviation equal to those of the s_i series

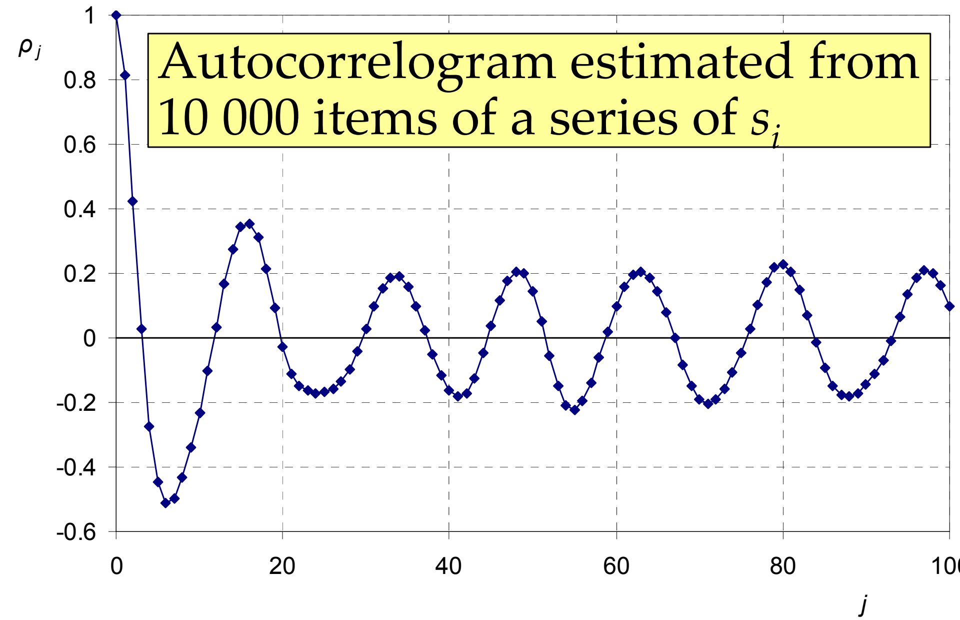


9. Quantification of antipersistence

- From the time series $x_i \equiv x_1(i)$ we construct time series of averages at several scales $\Delta = 2, 3, \ldots$, i.e. $x_\Delta(i) = (1/\Delta)\,[x_1(i\Delta - \Delta + 1) + \ldots + x_1(i\Delta)]$, and estimate the standard deviations σ_Δ.
- From a double logarithmic plot of σ_Δ *vs.* Δ we calculate the slope and then the *Hurst coefficient*: $H = 1 +$ slope.
- $H = 0.5$ indicates a purely random process;
- $H < 0.5$ indicates antipersistence;
- $H > 0.5$ indicates persistence.
- Here $H = 0.02$ (strong anti-persistence).
- H should be evaluated at large scales.
- Because any natural phenomenon has a positive autocorrelation at small scales and lags, a very mild slope is expected at small scales (as verified from the figure).
- Thus the *fractional Gaussian noise* model, which, by definition, yields constant slope for all scales, is not physically realistic for antipersistence.

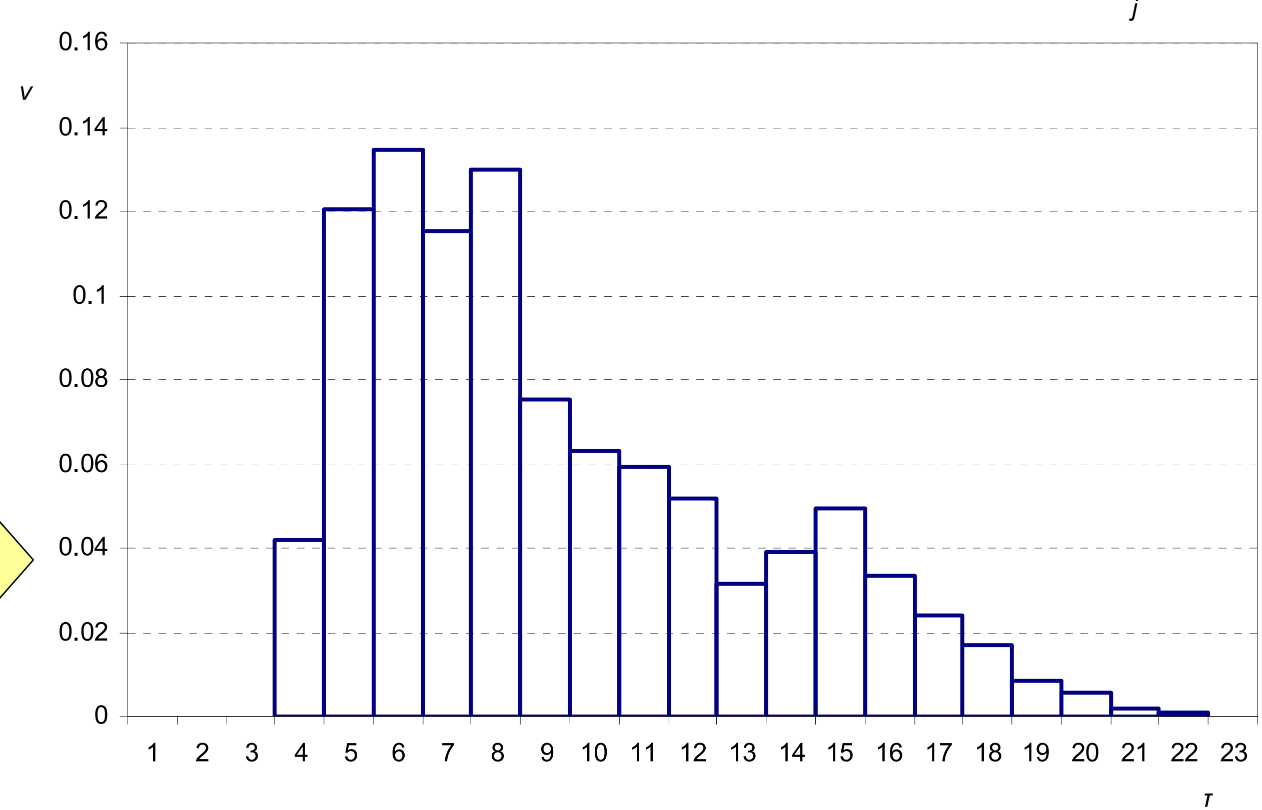


7. Is the system periodic?



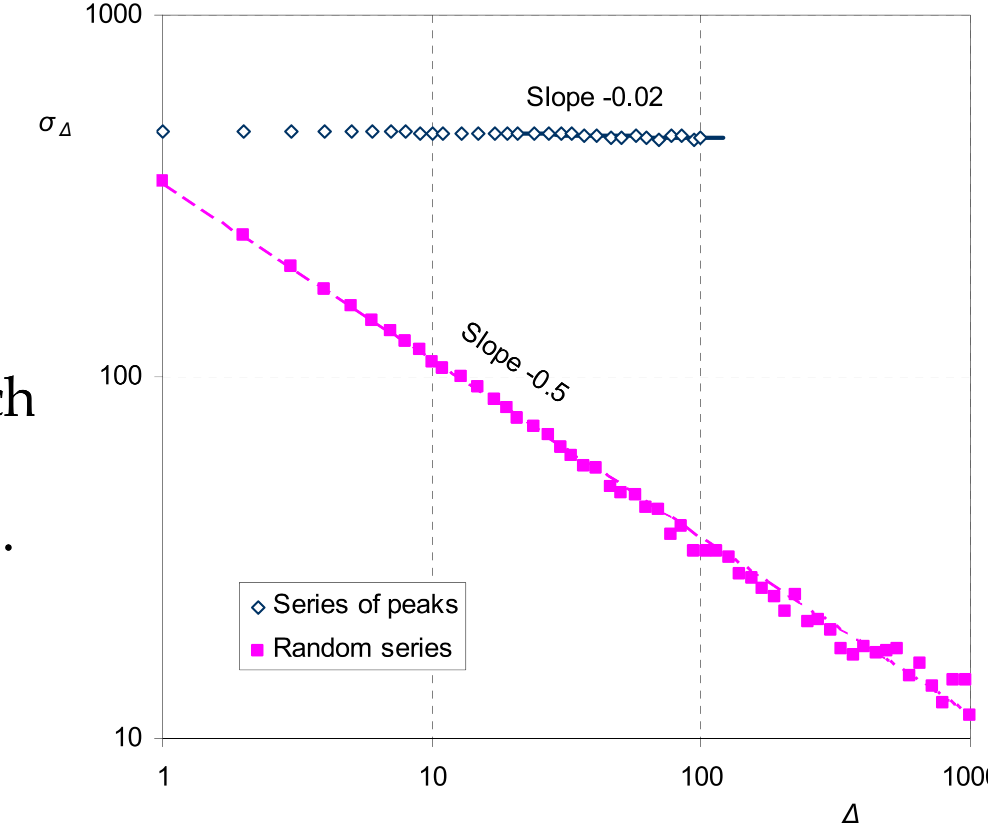
- From an inspection of either a time series plot (particularly when a short part of the series is studied) or the autocorrelogram, the system looks periodic.
- However, it is not periodic: the time τ between consecutive peaks is not constant.
- The system is characterized by **antipersistence**, which is different from periodicity.

Relative frequency ν of the time between consecutive peaks τ, estimated from 10 000 items of a series of s_i

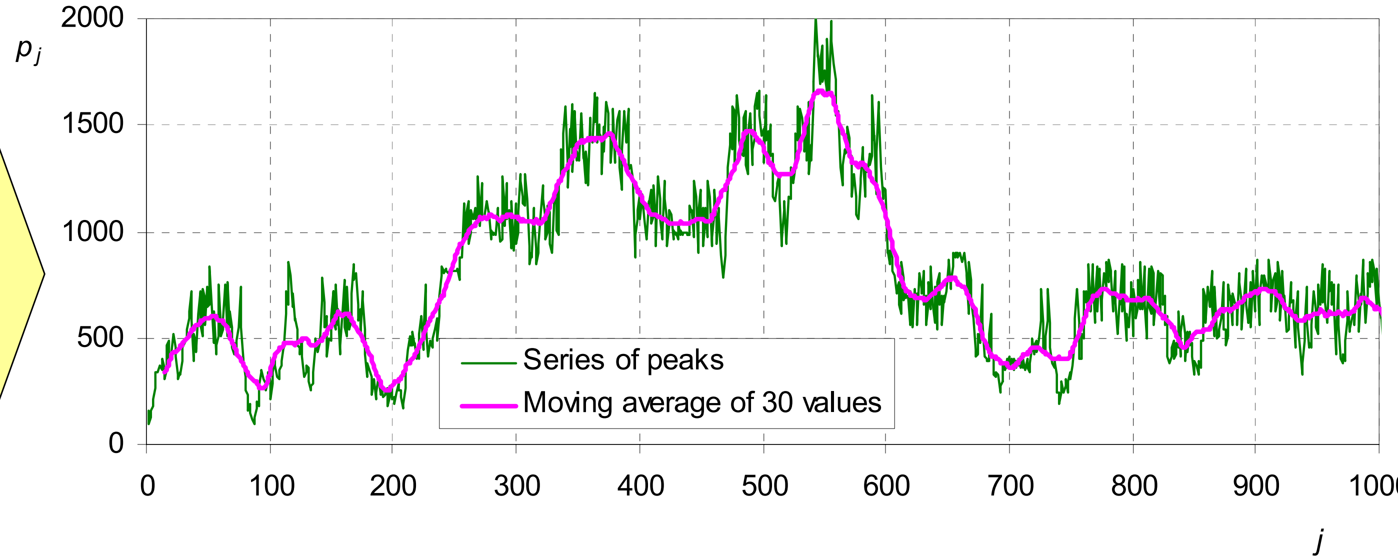


8. What is the behaviour of peaks?

Both the interarrival times of peaks (τ_j) and the storages at peaks (p_j), which are strongly correlated to each other (correlation coefficient 0.99), exhibit strong persistence ($H = 0.98$).



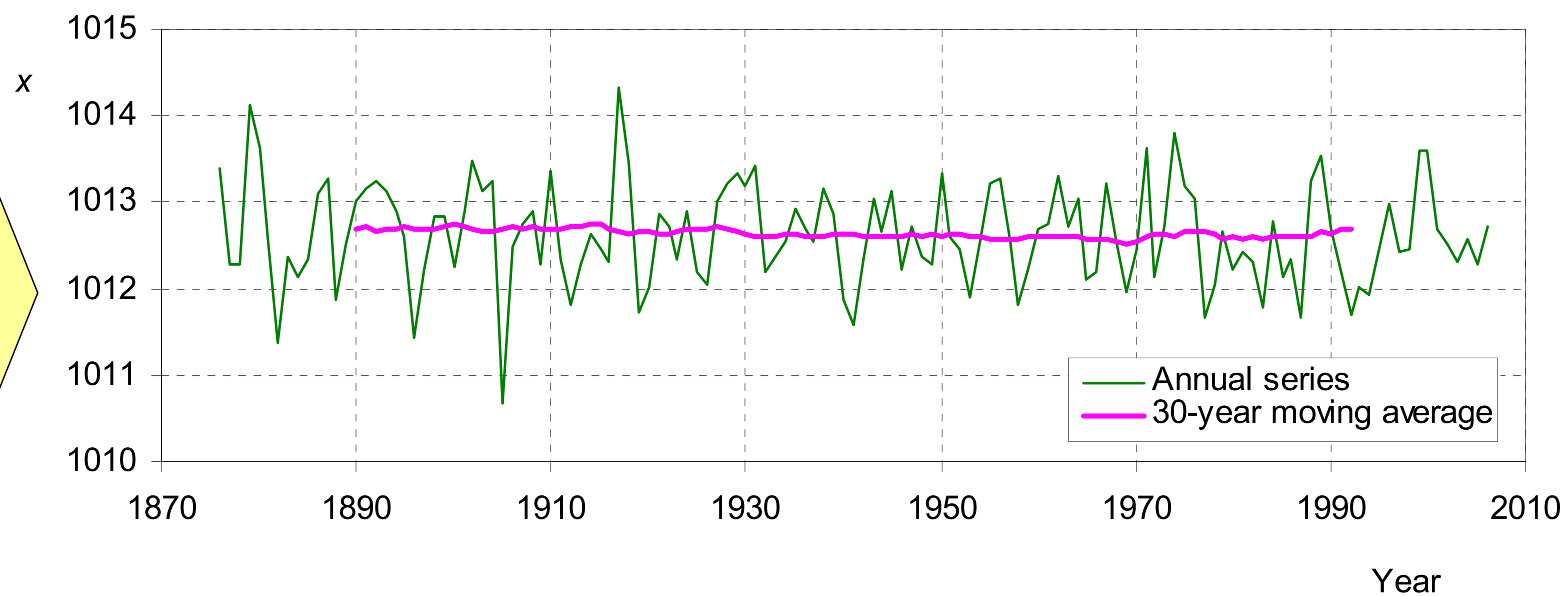
Series of soil water peaks p_j (in mm) extracted from a soil storage (s_i) series with length 10000; here j does not denote time but the rank of each peak in time order.



10. Are antipersistent process common in nature?

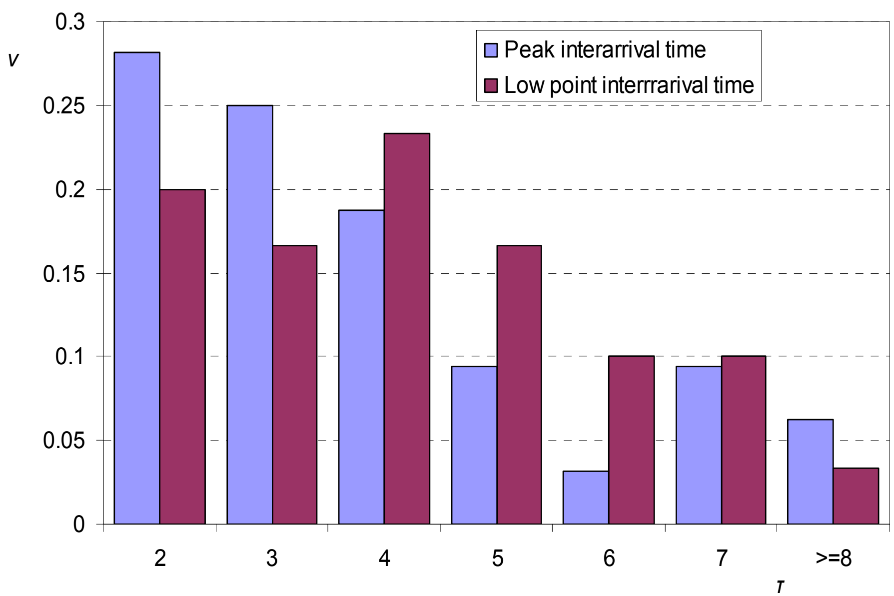
- It is not easy to find antipersistent real world phenomena (persistence is much more common), except in a few cases, commonly called "oscillations".
- The most widely known is the El Niño Southern Oscillation (ENSO), a fluctuation of air pressure and water temperature between the SE and SW Pacific.
- Typically it is quantified by the so-called Southern Oscillation Index (SOI), which expresses the difference in the air pressure between Tahiti (Polynesia) and Darwin (N. Australia); this difference is typically standardized in monthly scale by monthly mean and standard deviation.
- Here, instead of SOI, we have used the raw time series of the air pressure in Tahiti, to avoid the artificial effects of taking differences and standardizing.

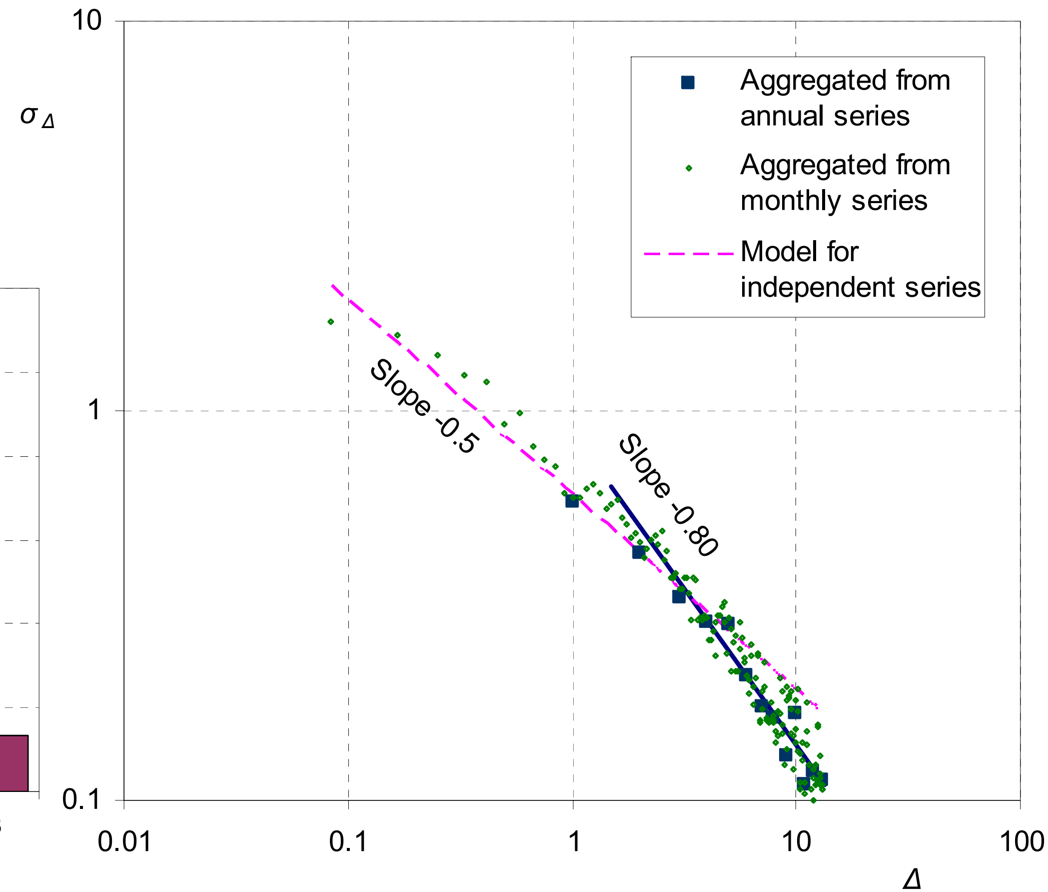
Time series of mean annual air pressure in Tahiti



11. General properties of the Tahiti air pressure

- The histogram of relative frequencies (ν) of the time between consecutive peaks or consecutive low points (τ), constructed for the Tahiti air pressure annual series, indicates that there is no periodic oscillation; the interarrival time varies between 2 and 13 years with an average of about 4 years.
- The plot of σ_Δ *vs.* Δ results in a Hurst coefficient: $H = 1 +$ slope $= 0.20$, which indicates antipersistence.





12. Conclusions and discussion

- Antipersistence is a property of a process, according to which the mean over a certain time scale tends to stabilize to the global mean faster than in a random process (the standard deviation of the time average tends to zero faster than in a random process).
- A typical representative of an antipersistent process is regarded to be the *fractional Gaussian noise* with Hurst coefficient $H < 0.5$; in this process autocorrelation coefficients for any lag are all negative, which makes the model inconsistent with natural processes.
- The toy model developed helps understand what a physically realistic antipersistent process is and inspect its basic characteristics.
- A physically realistic antipersistent process has positive autocorrelations for small lags (and small scales), which for larger lags alternate between negative and positive values.
- This property is common with periodic (cyclostationary) processes but the significant difference in antipersistence processes is that there is no constant period (e.g. between consecutive peaks or low points of the time series or of the autocorrelogram).
- On the other hand, the series of consecutive peaks may exhibit large and long excursions of local average from the overall mean, which indicates long-term persistence (sometimes incorrectly interpreted as nonstationarity).
- A physically consistent theory of antipersistent processes may assist in understanding and modelling of phenomena that have been called "oscillations“ such as the El Niño Southern Oscillation (ENSO); from the analysis of the Tahiti air pressure record, related to ENSO, it seems that it has the general properties of an antipersistent behaviour identified from the study of the toy model.

Acknowledgment: The time series of the air pressure in Tahiti is available online on a monthly scale at ftp://ftp.bom.gov.au/anon/home/ncc/www/sco/soi/tahitimslp.html