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On detectability of nonstationarity from data using statistical tools

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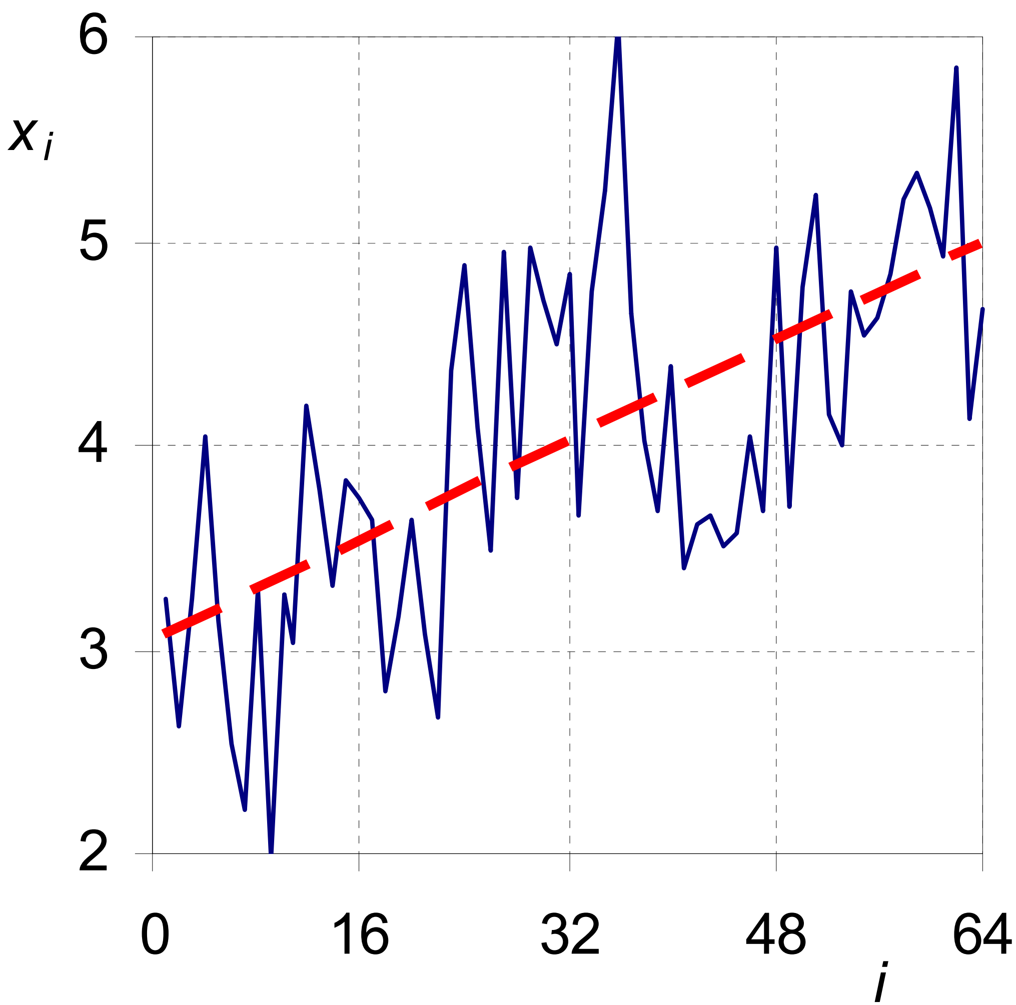
1. Abstract

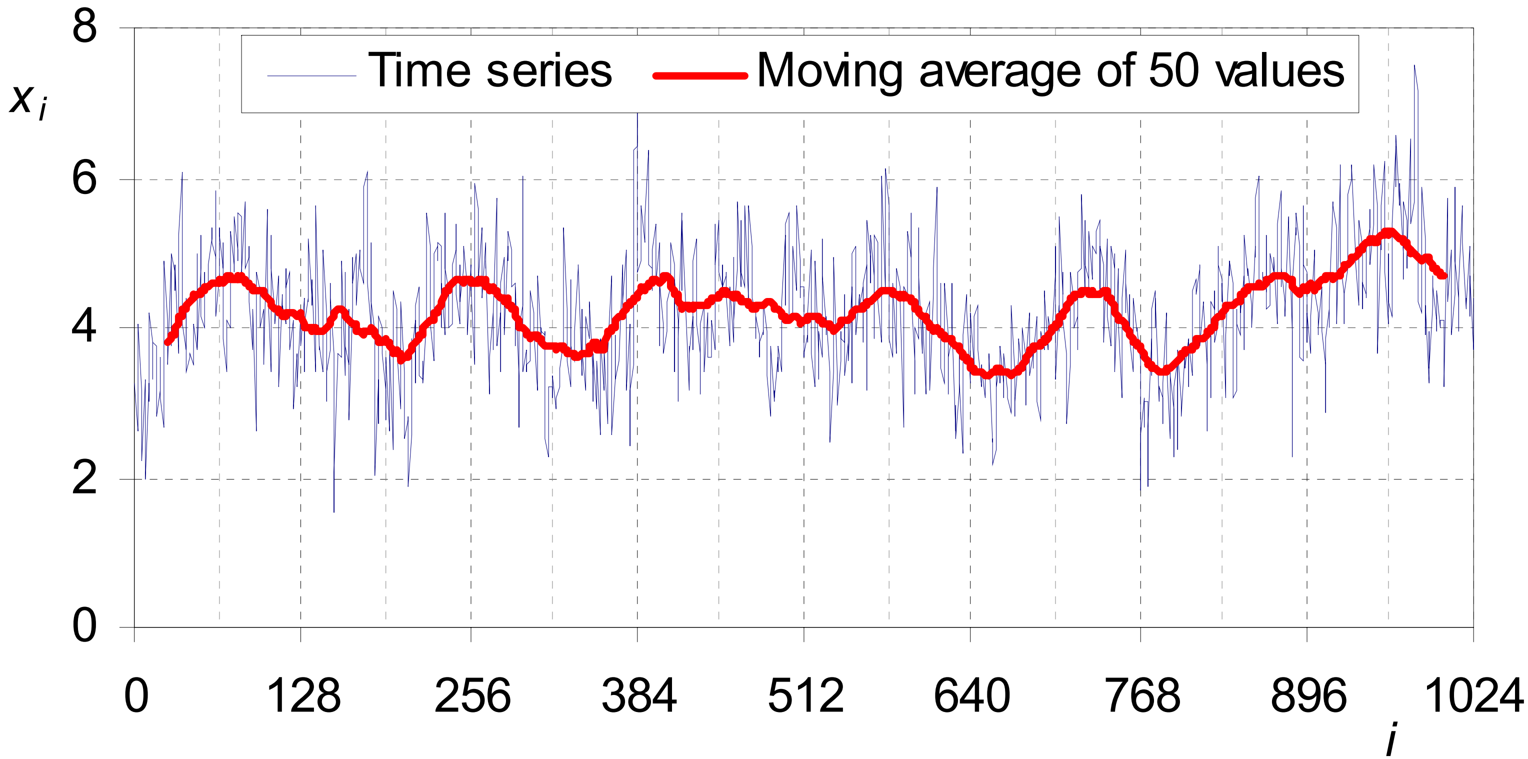
It has been a common practice in geophysical research to characterize observed time series as nonstationary and to apply statistical tools to detect nonstationarity. However, in many cases the logic of such detections is flawed, principally because stationarity and nonstationarity are not properties of the time series (phenomena) but of the mathematical processes (noumena) devised to model the phenomena, and also depend on our current knowledge of the system state. One of the most common flaws is the rejection of a stationarity hypothesis based on a classical statistical test which assumes that the process is independent in time, whilst it is well understandable that time independence is not an appropriate assumption for geophysical processes. In the case that a scaling behaviour is verified or assumed, one of the most common misuses of statistics is the characterization of a time series as nonstationary based on an estimate of a Hurst exponent greater than 1. Among the tools used for such estimations is the spectral representation of the time series. To demonstrate common flaws, several examples are synthesized, using data generated from hypothesized models, known a priori to be stationary or nonstationary. The examples aim to demonstrate that erroneous conclusions are very probable and to locate the origin of flawed results.

2. Is this time series nonstationary?



- Who could bet on "yes"?
- Note that the time series was generated from a simple scaling stochastic model (SSS) with Gaussian distribution, mean $\mu = 4$, standard deviation $\sigma = 1$ and Hurst coefficient $H = 0.9$ (the three-component generator in Koutsoyiannis, 2002, was used). Thus, the process x_t is **stationary**.
- However, the correlation coefficient of the time series with time is 0.65 (statistically significant according to classical statistics).

3. Does this time series contain trends?



- This is the same time series as before, extended over time (up to 1024).
- It was generated by a **stationary** SSS model.
- The term "trend" can be **acceptable** in a **loose setting**, to indicate that at some time a smoothed (e.g. time averaged) transformation of a time series increases or decreases.
- An attempt to use the term "trend" in a more **rigorous setting** (e.g. using mathematical functions) and combine it with **nonstationarity** may be **disastrous**.
- The fluctuations of the mean are common for an SSS process or, more generally, for a process with autocorrelation (Koutsoyiannis, 2006).

4. What is stationarity and nonstationarity?

- A stochastic process $X(t)$ is called *strict-sense* ***stationary*** if all its statistical properties are invariant to a shift of time origin: the distribution function of any order of $X(t + \tau)$ is identical to that of $X(t)$ (Papoulis, 1991).
- A process is called *wide-sense* ***stationary*** if its mean is constant and its autocovariance depends only on time differences, i.e. (Papoulis, 1991)

$$E[X(t)] = \mu, \quad E[(X(\tau) - \mu)\,(X(t + \tau) - \mu)] = C(\tau)$$

- A strict-sense stationary process is also wide-sense stationary (the inverse is not true).
- A process that is not stationary is called ***nonstationary***. In a nonstationary process one or more statistical properties depend on time.
- To characterise a process nonstationary, it suffices to show that some statistical property is a ***deterministic*** function of time.
- It has been a common practice to make such a characterization inspecting the data, i.e. detecting upward or downward "trends", fitting functions (e.g. linear expressions of time) to the trends and performing statistical tests to assess their significance.
- Such practices most of the times simply manifest **misuse of probability and statistics**, (Koutsoyiannis, 2006) because:
  - Stationarity and nonstationarity are properties of a process, not of time series (i.e. sample functions or series of observations).
  - A deterministic function is a function that can be produced by **deduction**, independently of the data (**a priori**; e.g. by a model that could predict them).
  - In contrast, according to this practice, the "trends" and "shifts" in the means are inferred by **induction** based on the data (**a posteriori**).
  - Hence such fitted lines are not "deterministic" and do not represent nonstationarity.

5. Are cumulative processes nonstationary?

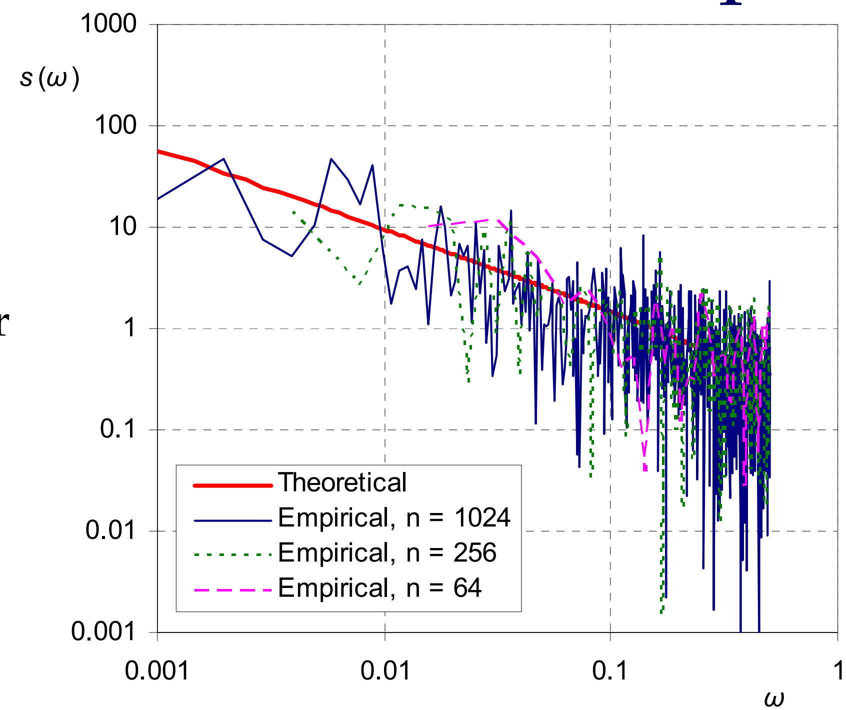
- A typical case of a nonstationary process is a **cumulative process** that in discrete time i can be expressed as $Y_i = Y_{i-1} + X_i$, where X_i is any stationary process, $i = 1, 2, \ldots$, and $Y_0 = 0$. Examples:
  - A random walk and a Wiener process in which consecutive X_i are independent (white noise) with zero mean ($\mu = 0$). The resulting mean of Y_i is $E[Y_i] = 0$ (not a function of time); yet they are nonstationary because $\mathrm{Var}[Y_i] \propto i$.
  - A Brownian motion, in which consecutive X_i are dependent with Markovian autocorrelation; for large time i, it has essentially the same properties with the Wiener process ($E[Y_i] = 0$, $\mathrm{Var}[Y_i] \propto i$).
  - A self-similar process, also known as the fractional Brownian motion, in which consecutive X_i are members of an SSS process ($E[Y_i] = 0$, $\mathrm{Var}[Y_i] \propto i^{2H}$, where H is the Hurst coefficient).
- Such cumulative process are abstract constructions, whose materialization can be done in several cases, i.e. in motion of molecules and in storing of inflows X_i in a reservoir; in the later case, the inflows are nonnegative so that $\mu := \mathrm{E}[X_i] > 0$; hence the mean of storage Y_i is proportional to time ($E[Y_i] = \mu\, i$).
- However, when these are materialize in real world processes, they change from nonstationary to stationary:
  - A Brownian motion occurs within boundaries (e.g. the glass containing water); bound Brownian motion is stationary (except in a transition period; Papoulis, 1991).
  - In a real world storage process, there are always some losses (e.g. evaporation, leakage, spills), so that the cumulative process should write $Y_i = aY_{i-1} + X_i$, where $0 < a < 1$ (with $(1 - a)Y_{i-1}$ representing the losses). It is easily proved that Y_i is a stationary process.
- Thus, **abstract cumulative processes** (without bounds and losses) are **nonstationary**, whereas **real world cumulative processes** (with bounds or losses) are **stationary**.

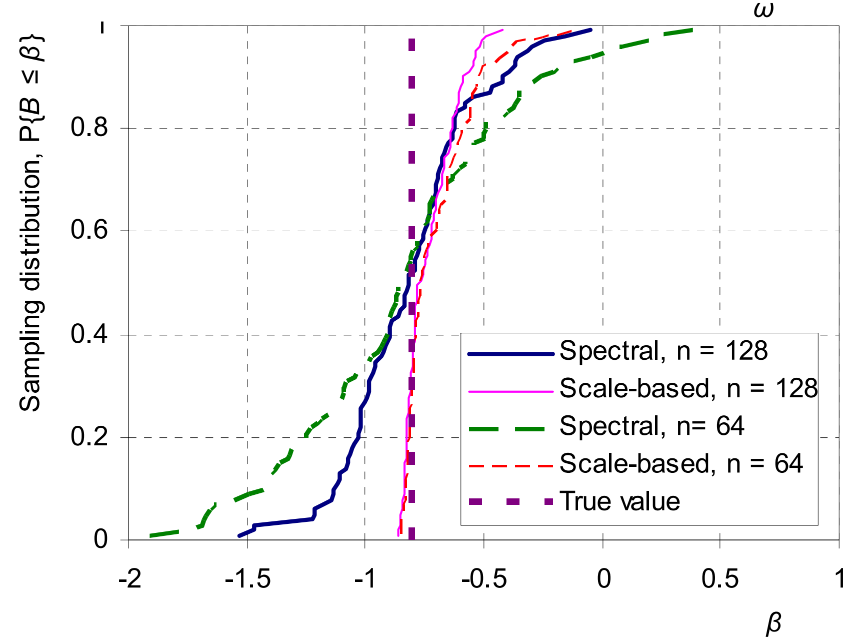
6. Can spectral methods detect nonstationarity?

- A common yet sophisticated method to detect nonstationarity has been based on the power spectrum (or spectral density) of a process.
- An SSS process (stationary) with Hurst coefficient H has a power spectrum $s(\omega) \propto \omega^{1-2H}$, where ω is the frequency. In a double logarithmic plot of s *vs.* ω, this results in a constant slope $\beta = 1 - 2H$ (for a purely random process, $H = 0.5$, $\beta = 0$; for a persistent process, $0.5 < H < 1$, $-1 < \beta < 0$; for an antipersistent process, $0 < H < 0.5$, $0 < \beta < 1$)
- In a nonstationary process (e.g. abstract cumulative), the power spectrum is clearly a function of time $s_i(\omega)$. Hence, estimation of a time-invariant power spectrum based on a time series x_i (e.g. using the periodogram) is not possible.
- However, such a nonstationary processes can be transformed to stationary by filtering it with an ideal bandpass filter (cutting frequencies below a specific ω_{L} or above a specific ω_{U}) and hence it can have a time invariant power spectrum (Keshner, 1982; Wornell, 1993).
- In this case, a stationarized self-similar process has a power spectrum $s(\omega) \propto \omega^{-1-2H}$; thus the slope in a double logarithmic of s *vs.* ω is $\beta = -1 - 2H$ (for a Wiener generating process, $H = 0.5$, $\beta = -2$; for a process with persistent intervals, $0.5 < H < 1$, $-3 < \beta < -2$; for a process with antipersistent intervals, $0 < H < 0.5$, $-2 < \beta < -1$).
- This result has been given the interpretation that a linear arrangement of the power spectrum in a double logarithmic plot with slope $\beta < -1$ manifests nonstationarity; this has been used as a technique to deem the time series that produces this power spectrum as nonstationary.
- However, this technique may yield erroneous results (particularly in short samples that are the rule in hydrology and geophysics) as demonstrated below using synthetic examples with a priori known properties.

7. False spectral estimation due to small sample

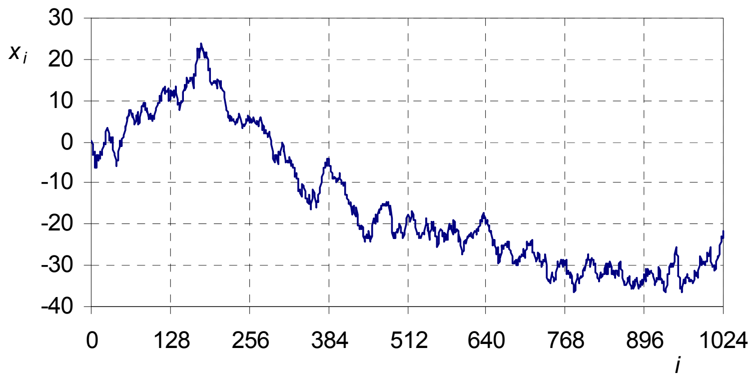
- Lengths of geophysical time series are usually small, and thus estimation of the slope in the power spectrum plot is highly uncertain.
- The upper figure demonstrates this estimation for the time series of panels 2-3 and for the indicated sample sizes n (the theoretical power spectrum is also known because the stochastic model is fully known).
- The slopes estimated from the empirical spectra for $n = 1024$, 256 and 64 are respectively -0.84, -0.90 and -1.66; the first two correspond to SSS (stationary) with $H = 0.88$, 0.95 (true value $H = 0.90$) whereas the last case erroneously deems the process as nonstationary with $H = 0.33$. (Note: here and in subsequent cases slopes are estimated for $\omega \leq 0.2$).
- The lower figure, depicting results of Monte Carlo simulations, shows that for sample sizes 64 and 128 the method erroneously deems the stationary process as nonstationary ($\beta < -1$) for 34% and 26% of the cases, respectively. A time scale-based method (using the standard deviation over several scales; Koutsoyiannis, 2003) is much more appropriate ($P\{B < -1\} = 0$).

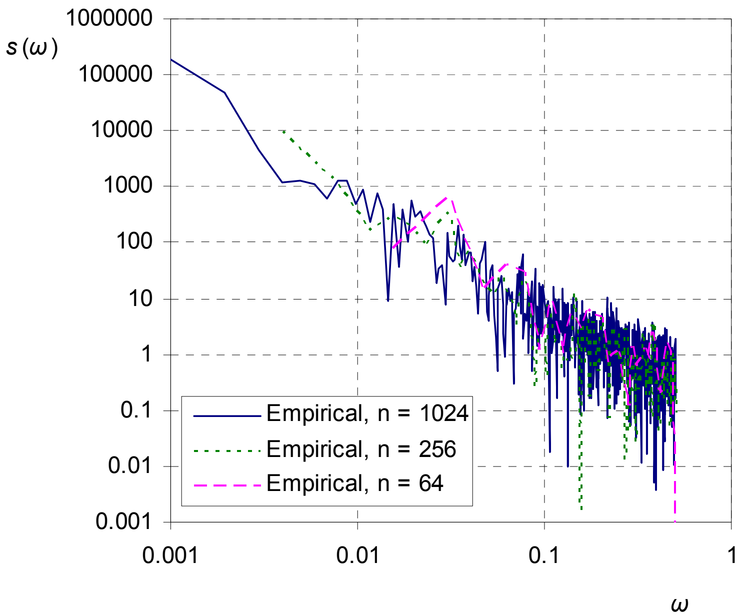




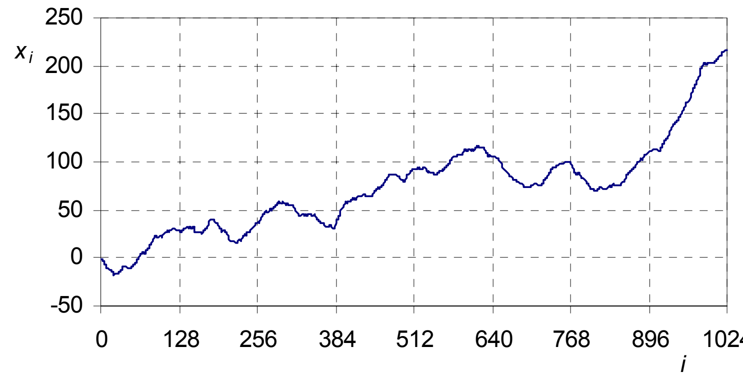
8. Spectral estimation for (abstract) cumulative processes

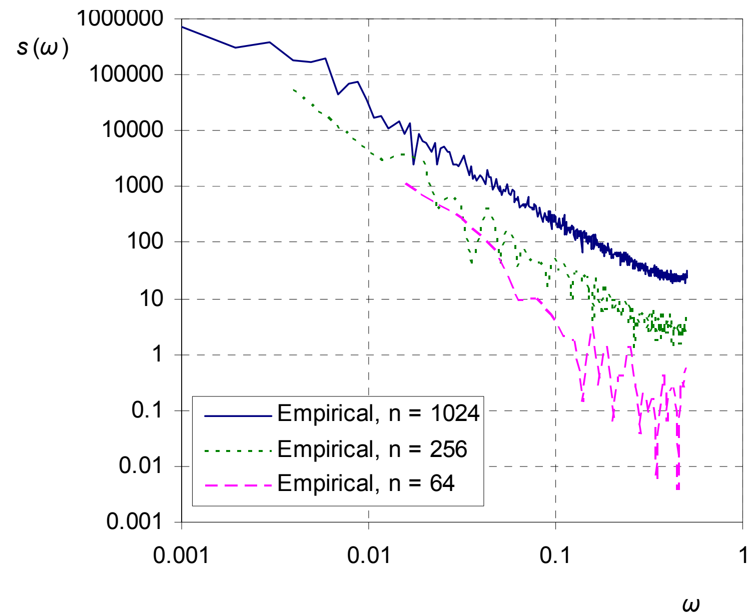
For a Wiener process (H = 0.50) the spectral method estimates slopes -2.11, -2.42 and -1.71 for n = 1024, 256 and 64, respectively. Thus the method correctly identifies the nonstationarity of the process and estimates H as 0.56, 0.71, 0.36, respectively (relatively close to the true value 0.50)





For the cumulative SSS time series of panels 2-3 (H = 0.90) after removal of the mean, the spectral method estimates slopes -1.98, -2.15 and -3.49 for n = 1024, 256 and 64, respectively. Thus, in the first two cases, the method correctly deems the processes as nonstationary but estimates incorrect H values (0.49 and 0.58, respectively). The third case (β = -3.49 does not have a meaning. (Notice the increase of $s(\omega)$ with n due to nonstationarity).





9. A cumulative processes of natural type (stationary)

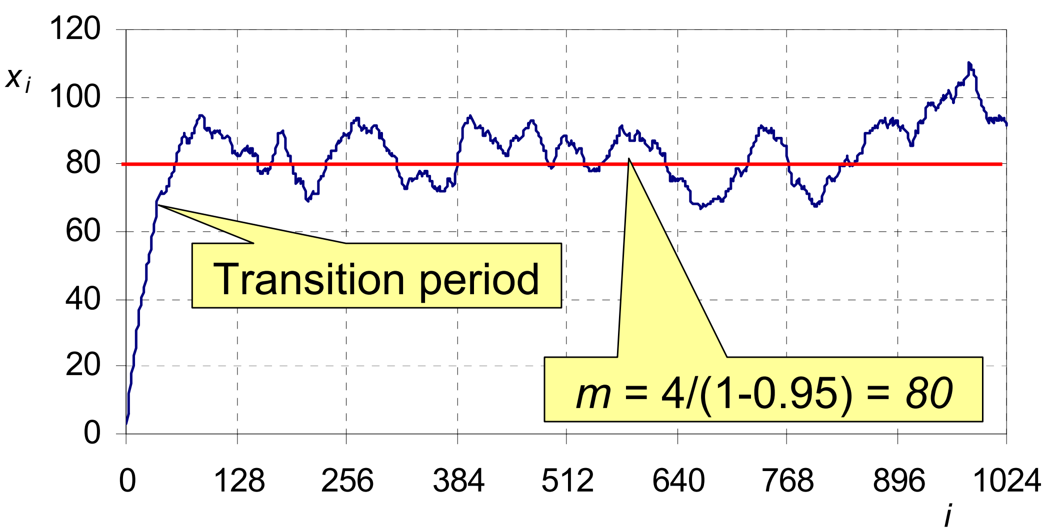
- A good example for a natural cumulative (storage) process is given by

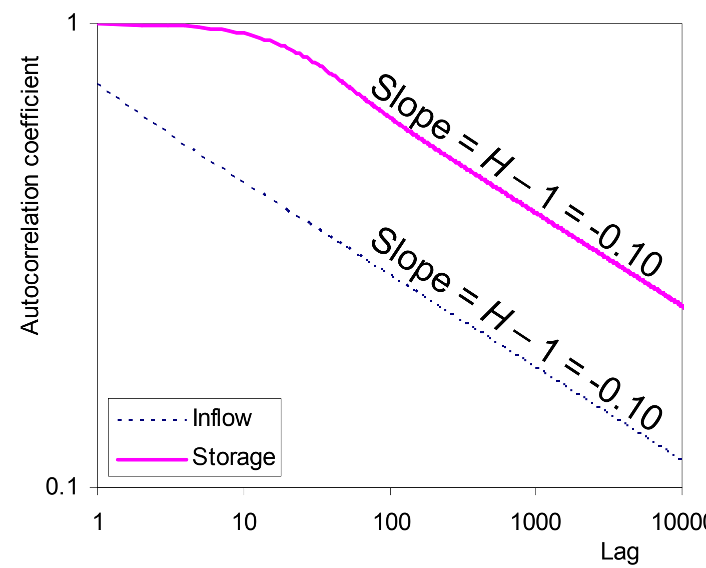
  $$Y_i = aY_{i-1} + X_i;\quad i = 1, 2, \ldots;\ Y_0 = 0$$

  where the inflows X_i can be assumed as SSS (stationary) and a is a constant slightly smaller than 1 (to account for losses). Clearly then, X_i is also stationary.
- Given the statistical characteristics of the process X_i, i.e. the mean μ, the standard deviation σ and the Hurst coefficient H, which fully describe the process autocovariance $\gamma_j := \mathrm{Cov}[X_i, X_{i+j}]$, the statistical characteristics of the process Y_i are easily derived as:

  $$m := E[Y_i] = \mu/(1 - a);\ g_0 := \mathrm{Var}[Y_i] = (b_0 + a\, b_1)/(1 - a^2);$$

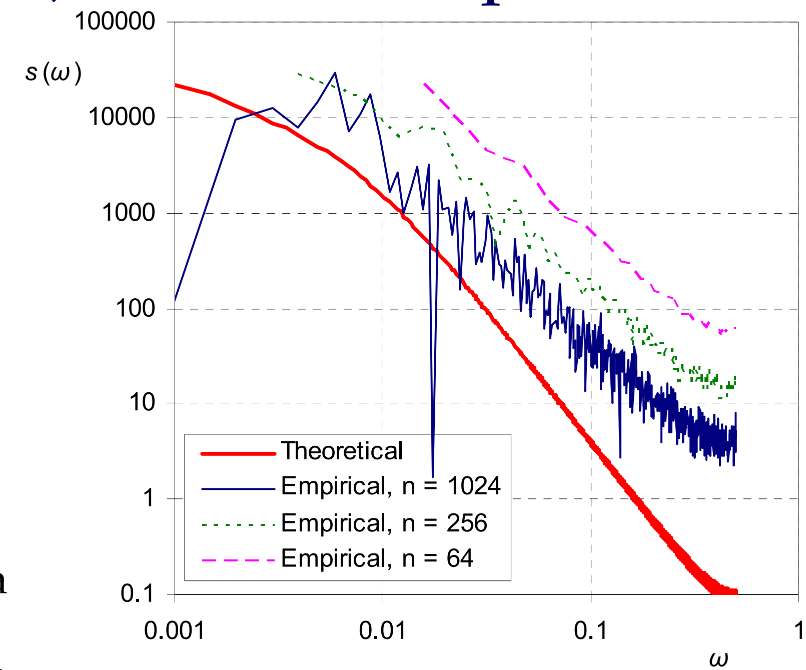
  $$g_j := \mathrm{Cov}[Y_i, Y_{i+j}] = b_j + a\, g_{j-1};\ \text{where } b_j := \mathrm{Cov}[Y_i, X_{i+j}] = \gamma_j + a\, b_{j+1}$$
- This allows theoretical derivation of the autocorrelation of Y_i (see figure on the right) and its power spectrum (see panel 10). The process Y_i is asymptotically scaling with same Hurst coefficient H.
- The figure on the left shows a time series of Y_i if the time series of X_i is that of panels 2-3 ($H = 0.80$, $\mu = 4$, $\sigma = 1$) and $a = 0.95$.

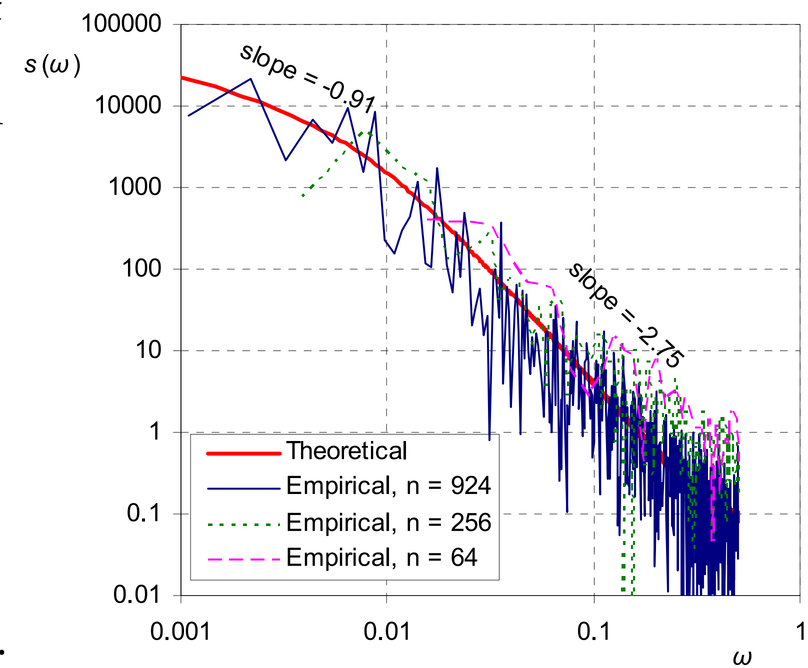




10. Spectral estimation for (natural) cumulative processes

- The upper figure shows the periodograms of the Y_i time series of panel 10 for the indicated sample sizes n, starting from the first point of the time series. The slopes are -1.63, -1.81, -1.89 for n = 1024, 256, 64, respectively. This would represent a nonstationary self similar process with H = 0.32, 0.41, 0.44, respectively. Such conclusions are totally incorrect because the process is stationary with H = 0.90.
- The lower figure shows the same periodograms but omitting the first 100 points to exclude the transition period. The slopes have become -2.35, -2.30, -2.09 for n = 924, 256, 64, respectively. These would represent a nonstationary self similar process with H = 0.68, 0.65, 0.54, respectively. The conclusions are again incorrect because the process is stationary with H = 0.90.
- The theoretical power spectrum is also shown in both figures. The slope of the main part is -2.75, indicating a nonstationary process with H = 0.88. This is incorrect (it would be almost correct if the parameter a in the model was 1 rather than 0.95). If we used the slope at the lower end of the power spectrum, i.e. -0.91, we would conclude that the process is stationary with H = 0.96. This is close to reality but cannot be seen unless the theoretical power spectrum is known (it cannot be derived from data).
- In conclusion, in all cases the empirical analyses lead to the incorrect conclusion of a nonstationary process.





11. Conclusions and discussion

- In the last years, statements such as "*Physical processes are mostly nonstationary*" have become very common and widely accepted.
- Such statements do not respect the definition of nonstationarity, according to which observed changes should be deterministic functions of time; deterministic functions should be be produced by deduction, independently of the data, and not by induction, based on the data.
- Furthermore, such statements are inconsistent with the fact that stationarity and nonstationarity are properties of a stochastic model (a mathematical model implying an ensemble of an unlimited number of potential realizations) and not of a natural time series (a unique realization of a physical process).
- A more consistent replacement of this type of statements would be "*Physical processes are different from purely random processes and even from Markovian processes*".
- One of the more sophisticated tools to detect nonstationarity has been the spectral analysis, according to which a possible linear arrangement of the power spectrum in a double logarithmic plot with slope $\beta < -1$ manifests nonstationarity.
- It is demonstrated here that slopes β steeper than –1 can emerge for processes that are a priori known to be stationary, either because of sampling inaccuracy (for small sample sizes) or because of high and complex dependence (autocorrelation) of the process, even if the number of points is large.
- Hence, the stationarity or nonstationarity of a process should be studied using reasoning rather than processing data.

12. References

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"He who loves practice without theory is like the sailor who boards ship without a rudder and compass and never knows where he may cast"

Leonardo da Vinci