European Geosciences Union General Assembly 2010
Vienna, Austria, 2-7 May 2010
Session HS5.5: Stochastics in hydrometeorological processes: from point to global spatial scales and from minute to climatic time scales

Three dimensional Hurst-Kolmogorov (HK) process for modelling rainfall fields

P. Dimitriadis, D. Koutsoyiannis and A. Paschalis
Department of Water Resources and Environmental Engineering
National Technical University of Athens
(www.itia.ntua.gr)

1. Abstract

A three-dimensional (3D) stochastic simulation model is presented, which is a direct extension of the 1D simple scaling process (fractional Gaussian noise). The 3D process can generate time-varying 2D rainfall fields through a rather simple procedure, as well as other time-varying 2D spatial geophysical fields, consistent with the observed 2D long-term spatial persistence over time (3D slowly decaying autocorrelation over scale). Moreover, the differences between 1D (generating rainfall time series at a point), 2D (generating rainfall fields for specific time steps) and 3D (generating spatio-temporal rainfall fields) scaling processes are also being investigated through some applications based on observed rainfall fields.

2. Hurst phenomenon and the HK process

"High tendency of high/low values to occur in natural events": Hurst (1951) ➔ Slowly decaying autocorrelation over scale➔ Power-law behavior (Kolmogorov, 1940).

$$\left(\underline{Z}_v^{(k)} - \mu\right) =_d \left(\frac{k}{l}\right)^A \left(\underline{Z}_v^{(l)} - \mu\right) \quad \begin{cases} \mu = E[\underline{Z}_v] \\ A = D(1 - H) \\ \underline{Z}_v^{(k)} = \dfrac{1}{k^D} \sum_{i_1=(v_1-1)k+1}^{v_1 k} \cdots \sum_{i_D=(v_D-1)k}^{v_D k} \underline{Z}_{i_1,\ldots,i_D} \end{cases}$$

- $\underline{Z}$: random field of interest (assumed stationary and isotropic)
- $\underline{Z}_v$: mean aggregated field (at a spatio-temporal scale)
- v: vector index of random field indicating location in the field
- k,l: any aggregated scales of the process
- μ: mean of the process
- $=_d$: equal in distribution function
- A: power law exponent of autocorrelation over scale
- D: dimension of vector index space of random field (v)

3. Hurst coefficient (H) of the HK process

HK process depends on the characteristic parameter $0<H<1$. Here, the estimation of the H coefficient is done via the minimization of the square error (SE_H) of the empirical ($S^{(k)}$) and true ($\gamma^{(k)}$) variance over scale k of the process. A method of Koutsoyiannis (2003) for the estimation of H was extended to the D-dimensional process.

$0 < H < 0.5 \rightarrow$ Anticorrelated $(\rho < 0)$

$H = 0.5 \rightarrow$ Independent $(\rho = 0)$

$0.5 < H < 1 \rightarrow$ Correlated $(\rho > 0)$

$$\tilde{S}^{(k)} = R S^{(k)}, \; R(k;H) = \frac{N/k^D - 1}{N/k^D - \left(N/k^D\right)^{2H-1}}$$

$$SE_H = \sum_{k=1}^{k'} \left[ln(\tilde{S}^{(k)}) - ln(\gamma^{(k)}) \right]^2 / k^p, \; p = 2$$

4. Field Normalization

HK process generates random fields that follow the $N(0,1)$. Here the following transformation (Papalexiou et al., 2007) is used, where its coefficients p_i are estimated through the minimization of the square error of the model and $N(0,1)$ distribution function.

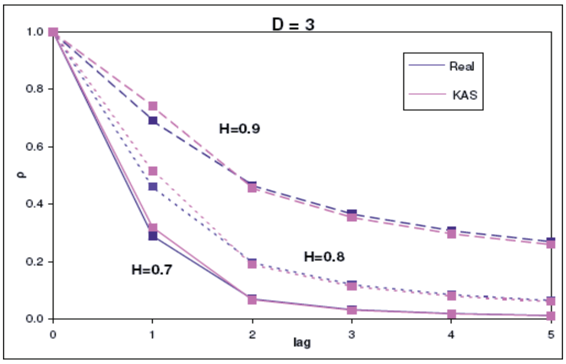
$$Z_N = \left(p_1 Z^{-p_5} + p_2\right)\left(p_3 + \sqrt{(1 + 1/p_4)\ln\left[p_4 (Z - p_3)^2 + 1\right]}\right)$$

5. Autocorrelation function of the HK process

The autocorrelation function ρ (acf) of HK process is independent of the aggregated scale k. For any displacement vector r (lag) is expressed by a power-law function. The latter can be integrated over a discrete space of points P_v and $P_{v'}$, so that the discrete spatio-temporal acf can be estimated. This integration has an analytical solution for D=1. Koutsoyiannis et al. (2010) proposed an approximated solution for D=2. Here, this solution is extended for D=3 (referred as KAS).

$$\rho_r^{(k)} = \rho_r = \gamma_r^{(k)} / \gamma_0^{(k)} \text{ and } \rho_r = C_D |r|^{-B} \rightarrow \rho_r = \int_{P_v}\int_{P_{v'}} C_D |r|^{-B} dv\, dv' \text{, where :}$$

$$\gamma_0^{(k)} = k^{-B}\gamma_0^{(1)} \;, r = \|P_v - P_{v'}\| \;, B = 2D(1-H) = 2A \;, C_D = \frac{(2H-1)[D(2H-1)+1]}{(D+1)}$$



➢ KAS for $D \le 3$

$$\rho_{D,r} = \min\left\{C_D\left(\rho_{1,r}/C_1\right)^D, \rho_{1,r}\right\}$$

$$\rho_{1,r} = |r+1|^{2H}/2 + |r-1|^{2H}/2 - 2|r|^{2H}$$

6. Simulation scheme for generating HK process

SMA stands for Symmetric Moving Average and it can be used to generate a stochastic process with any structure of autocorrelation or power spectrum (Koutsoyiannis, 2000). Here, the SMA scheme has been extended to three spatio-temporal dimensions (direct extension from 1D and 2D schemes).

- Z_v: generated random variable of interest
$$Z_v = \sum_{y=\underbrace{-q,-q,\ldots,-q}_{D}}^{q,q,\ldots,q} \alpha_y W_{v-y}$$

- W: discrete white noise (random field) with zero mean ($\mu_w = 0$) and unit standard deviation ($\sigma_w = 1$) (since $\underline{Z}$ has been normalized).

- q: finite limit for the range of coefficients α_y (for m, the desired number of autocorrelation coefficients that are to be preserved, and for $\beta\alpha_0$, the acceptance tolerance).

- α_y: field of coefficients that can be determined through the Fourier transform F_γ of the autocovariance field γ_Z (Koutsoyiannis, 2000, Koutsoyiannis et al. 2010).

7. Spectral density and α_y coefficients of SMA

The spectral density F_γ of the stochastic field can be determined via the Fourier transform of the discrete form of autocovariance γ_r . It can be shown that the Fourier transform F_α, of the field α_y, is related to F_γ (for q=∞), thus the α_y field can then be estimated.

$$F_\gamma = 2\pi\int_0^\infty |r|^{D-1}\gamma(r)J_0(2\pi qr)dr = 2\pi C\int_0^\infty |r|^{D-B-1} J_0(2\pi qr)dr \rightarrow$$

$$\rightarrow F_\gamma = g_\alpha |q|^{B-D}, g_\alpha = \gamma_0 CE, \text{where } E = \pi^{B-D+1}\frac{\Gamma\left[(D-B)/2\right]}{\Gamma\left[(B-D+2)/2\right]} \quad \text{for } B > D-3/2$$

$$F_\alpha = \sqrt{F_\gamma} \rightarrow \alpha_y = \alpha_0\rho(\|y\|;H'), \text{where } B'-D = (B-D)/2 \rightarrow H' = H/2+1/4$$

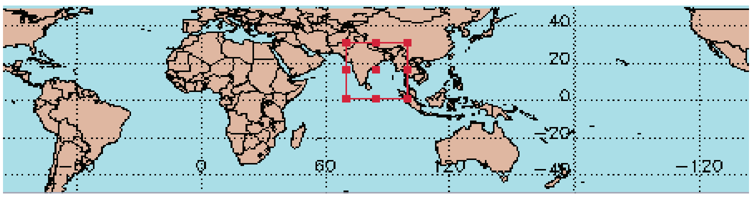
$$\gamma_0 = \sum_{y=\underbrace{-q,-q,\dots,-q}_{D}}^{q,q,\dots,q} \alpha_y^{\,2} \rightarrow \alpha_0^{\,2} = \gamma_0 / \sum_{y=\underbrace{-q,-q,\dots,-q}_{D}}^{q,q,\dots,q} \rho^2(\|y\|;H')$$

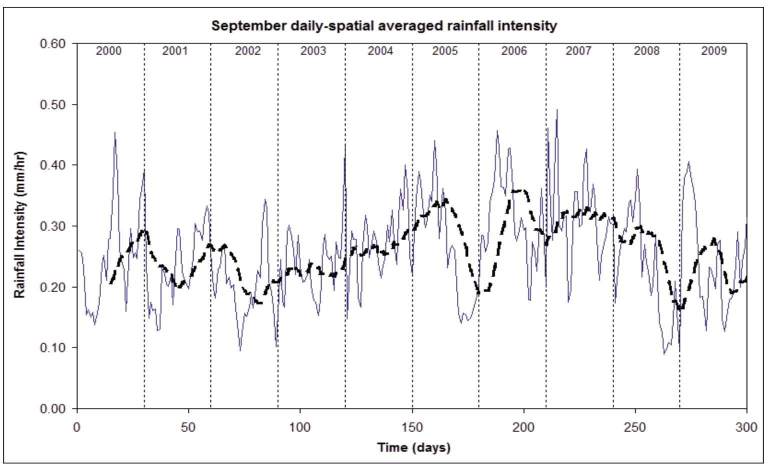
$$g_\alpha = \sqrt{g_\gamma} \rightarrow \alpha_{0,q=\infty} = \frac{\sqrt{\gamma_0 C(H)E(H)}}{C(H')E(H')}, \text{for } B > D-3/2$$

8. Case study on observed rainfall fields

The application presented is based on an observed rainfall field in the India Ocean SE of India (coordinates: 30N-0N, 70E-100E). The data were acquired from TRMM NASA satellite system (available on-line):
http://disc2.nascom.nasa.gov/Giovanni/tovas/TRMM_V6.3B42.shtml

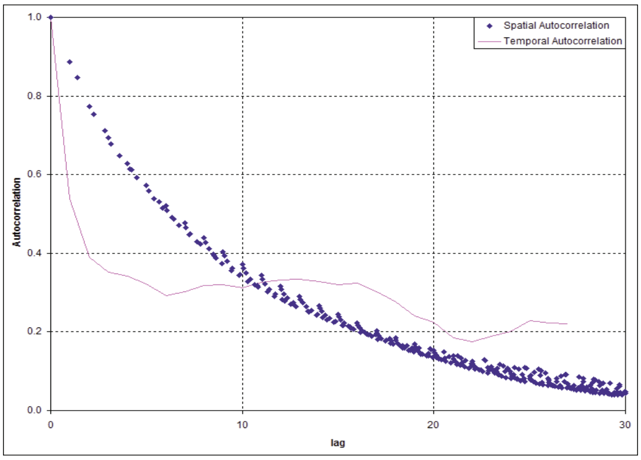
The sample consists of a spatial grid 121x121 points (of a 0.25° x 0.25° spatial resolution, approx. 25 km x 25 km) and a temporal 30-days grid. The latter grid was created with a 10-years daily average of the month with the highest spatial averaged accumulated rainfall over the area between 2000 and 2009 (September).

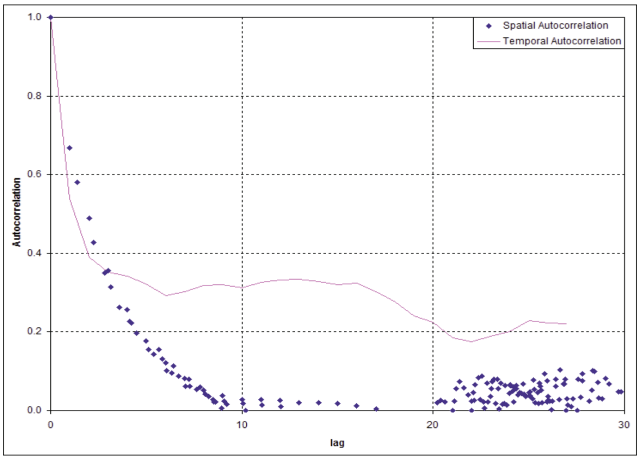




9. Spatial - temporal resolution correspondence

The spatial and temporal displacements are physically different. For the assumption of isotropy to be valid spatial and temporal acf should match at least for the first lags. Thus, by constructing the diagrams of spatial and temporal acfs, the resolution correspondence can be evaluated.
At the left figure it can be seen that the space is isotropic. The spatial resolution is estimated to 25% of the initial unit (25km), so that the two acfs decay at the same rate (right figure). Thus the spatial grid is changed to 31x31 points (more coarse).

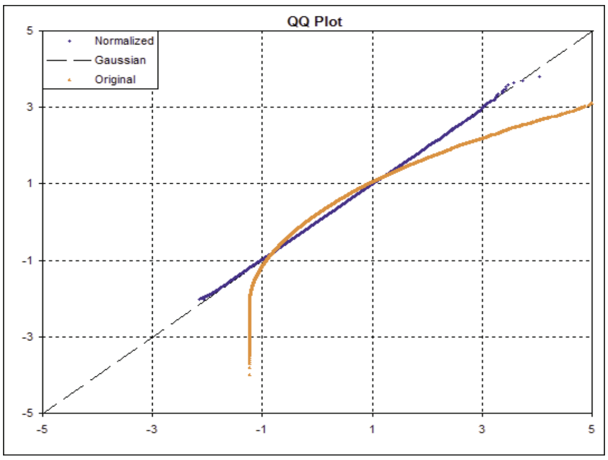




10. Normalization of rainfall field

As described in section 4, the transformation by Papalexiou et al. (2007) is used to normalize the observed rainfall field. The zero values of the natural field are replaced with the small value of 1e-5.

The simulated field should be converted to natural units by solving arithmetically the inverted transformation.

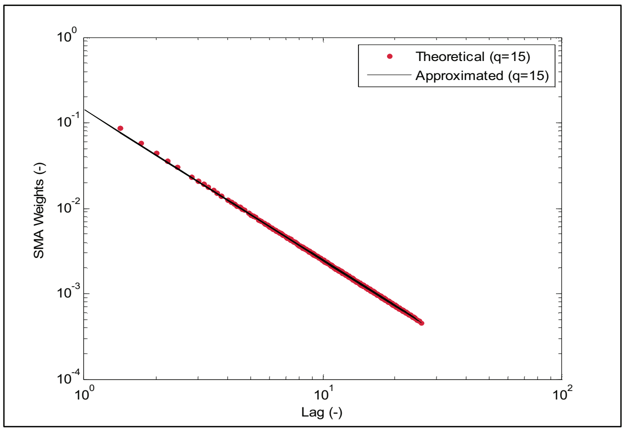


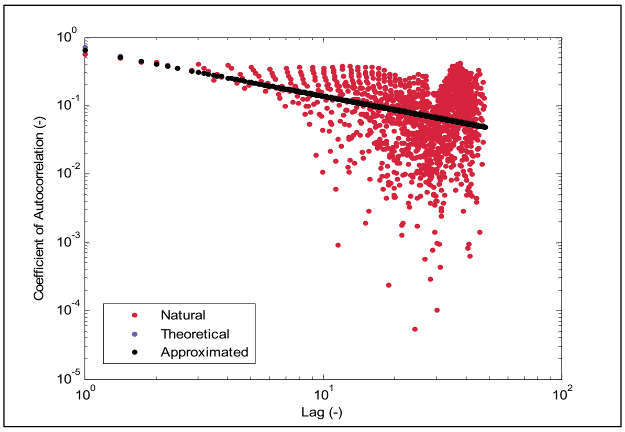
$p_1 = 9.0$, $p_2 = -0.6$, $p_3 = -1.8$, $p_4 = 2.8$, $p_5 = 7.0$, $SE = 14.0$

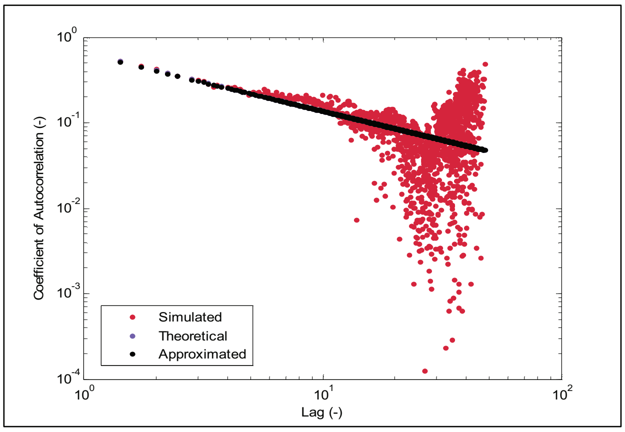
The table below shows several statistical parameters of the case study for a random chosen simulation run.

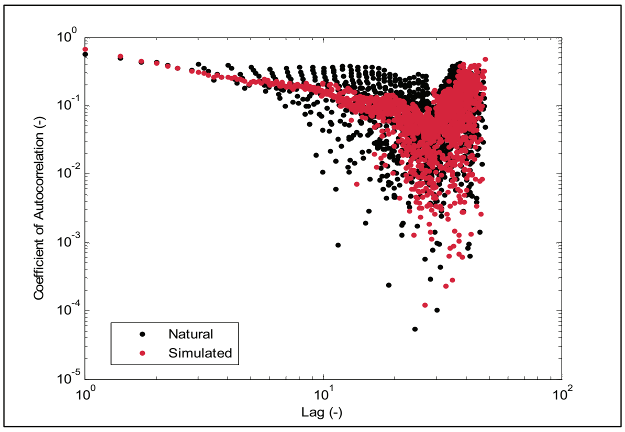
	Natural	Normalized	Simulated
Mean	0.25	0.00	0.00
Standard Deviation	0.20	1.00	0.94
Skewness	1.39	0.12	0.55
H biased	-	0.89	0.86
H unbiased	-	0.91	0.90

11. Stochastic simulation model (figures)

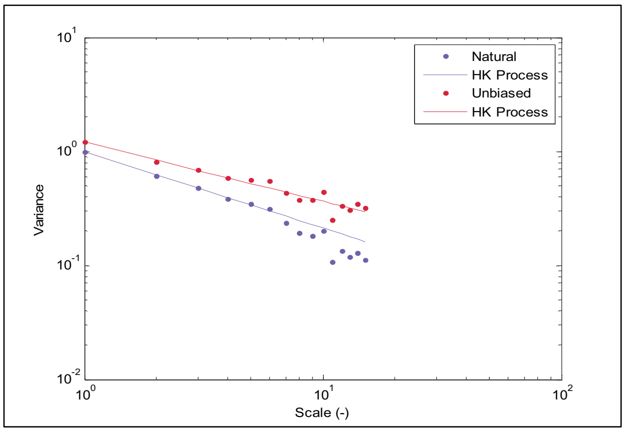


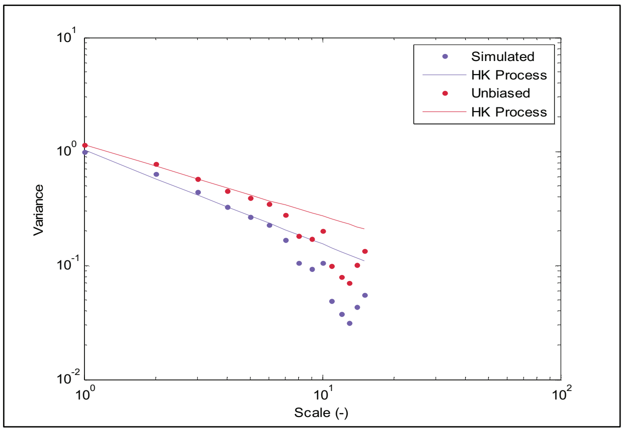


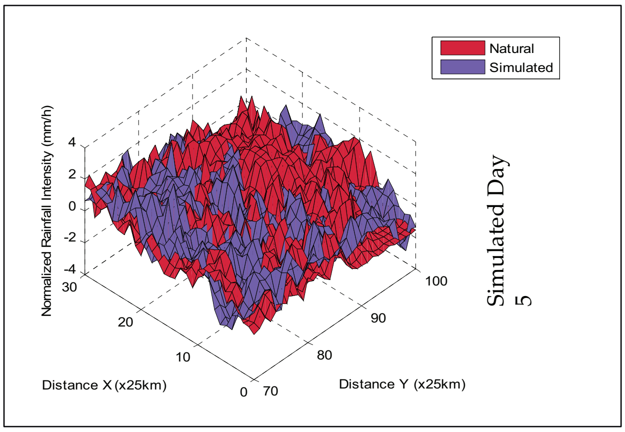


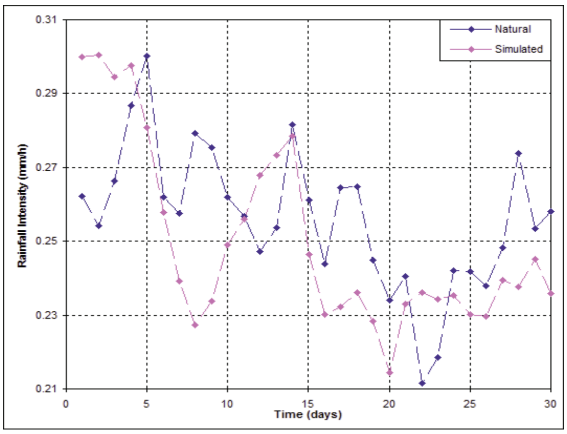


12. Stochastic simulation model (figures)









13. Conclusions

- A three-dimensional (3D) stochastic simulation model is proposed, which is a direct extension of the 1D simple scaling process (HK or FGN), that can generate time-varying 2D spatial geophysical fields consistent with 2D long-term spatial persistence over time (3D slowly decaying autocorrelation over scale).
- All of the framework used at the HK and SMA process is extended for any other dimension D of the field. The autocorrelation function over spatio-temporal displacement can be adequately approximated with the extended Koutsoyiannis et al. (2010) solution for D≤3.
- The 3D model is verified through an application based on an observed rainfall field.

References

Hurst, H.E. (1951) Long term storage capacities of reservoirs. Trans. ASCE 116, 776-808.

Kolmogorov, A.N., WienerscheSpiralenund einigeandereinteressanteKurvenin HilbertschenRaum, Dokl. Akad. NaukURSS, 26, 115–118, 1940.

Koutsoyiannis, D. (2000) A generalized mathematical framework for stochastic simulation and forecast of hydrologic time series. Wat. Resour. Re. 36(6), 1519-1534.

Koutsoyiannis, D. (2003), Climate change, the Hurst phenomenon, and hydrological statistics, Hydrological Sciences Journal, 48(1), 3-24.

Koutsoyiannis et al., (2010), Two-dimensional Hurst-Kolmogorov dynamics and its application to the study of rainfall fields (in preparation).

Papalexiou, S.M., A. Montanari, and D. Koutsoyiannis, Scaling properties of fine resolution point rainfall and inferences for its stochastic modelling, European Geosciences Union General Assembly 2007, Geophysical Research Abstracts, Vol. 9, Vienna, 11253, European Geosciences Union, 2007.