Multifractality: at least three moments !

D. Schertzer[1], I. Tchiguirinskaia[1], and S. Lovejoy[2]

[1]Université Paris-Est, Ecole des Ponts ParisTech, LEESU, Marne-la-Vallée, France
[2]McGill U., Physics dept, Montreal, Canada

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1 Summary of our comment

Multifractality of rainfall has been the central topic of theoretical hydro-meteorology for almost three decades and it is therefore important to improve the parameter estimation techniques of multifractal fields. Unfortunately, we believe that the present paper (Lombardo et al., 2013) brings more confusion than clarification on this issue. First, the conclusions of this paper, as well as part of its title, are misleading. The authors indeed claim that only the first and second statistical moments of a multifractal field are safely estimated. Unfortunately, these two moments are insufficient to determine the nonlinear scaling moment function $K(q)$[1] of a multifractal field R, e.g. the rain rate, observed at various resolutions[2] $\lambda = L/l$ (where L is the largest scale, l is the observation scale) and where $< . >$ denotes a given ensemble average:

$$< R_\lambda > \approx \lambda^{K(q)} \tag{1}$$

Indeed, contrary to the linear case of a scaling moment function $K(q)$ of a uni/monofractal field, a third independent value is at least required to estimate the curvature of $K(q)$ for multifractal fields. Therefore, their conclusions would mean that the multifractal parameters could not be safely assessed and multifractals would be therefore of little interest to simulate rainfall.

Secondly, the present paper fully ignores the concept of second order multifractal phase transition, introduced years ago (Schertzer and Lovejoy, 1992, Schertzer et al., 1993, Schertzer and Lovejoy, 1994) (for recent reviews: Schertzer et al. (2010), Schertzer and Lovejoy (2011), Lovejoy and Schertzer (2013). This phase transition explains not only in a straightforward manner the qualitative observations made by the authors that the estimates of the statistical moments of order $q \geq 3$ of their numerically simulated

[1]which has been often denoted $-\xi(q)$ for structure functions.

[2]these observations could be obtained by various means, e.g.: aggregation, fluctuation estimates.

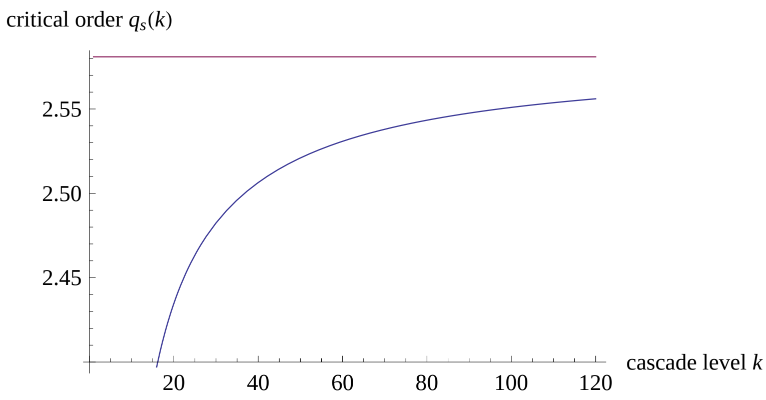

Figure 1: Graph of the theoretical, critical order $q_s(k)$, above which the estimate over a sample of a statical moment are spurious, at the k^{th} level of the cascade ($k = 1, 120$) and for the set of parameters chosen by the authors, as well as its asymptotic value $q_s(\infty) \approx 2.582$.

cascades seem to be spurious, but provides rigorous, analytical results. Indeed, this phase transition occurs at a critical moment order q_s that is analytically defined from $K(q)$ and such that the estimates over a sample of all the statistical moments of order $q > q_s$ are spurious. We show below that the discrete cascade model (Lombardo et al., 2012) used by the authors yields a theoretical critical order $q_s(k)$ that depends on the k^{th} level of the cascade. Its graph is displayed in Fig.1 for the parameter set chosen by the authors, but with a much larger number of steps to show its slow convergence to $q_s(\infty) \approx 2.582$. Because this asymptotic value is also an upper bound, the estimates of the moments of order $q > q_s(\infty)$ are all spurious. The simulations and qualitative observations of the present paper can be therefore seen as illustrations of this phase transition rather than new findings.

2 Multifractals and phase transitions

To go beyond the straightforward counter-argument that we can learn a lot on the curvature of $K(q)$ from statistical moments of non integer orders q's, let us recall that a convenient and physically meaningful choice of three independent multifractal parameters correspond to:

- the (Hurst) scaling exponent $H = -K(1)$ of the mean field. The value $H = 0$ corresponds to a 'conservative field", i.e. a field whose mean is strictly scale invariant, whereas $H \neq 0$ rather corresponds to a fractional integration/derivation of a conservative field;
- the mean intermittency measured by the codimension $C_1 = dK(q)/dq|_{q=1} + H$ of the support of the mean field; its lower bound $C_1 = 0$ corresponds to a homogeneous

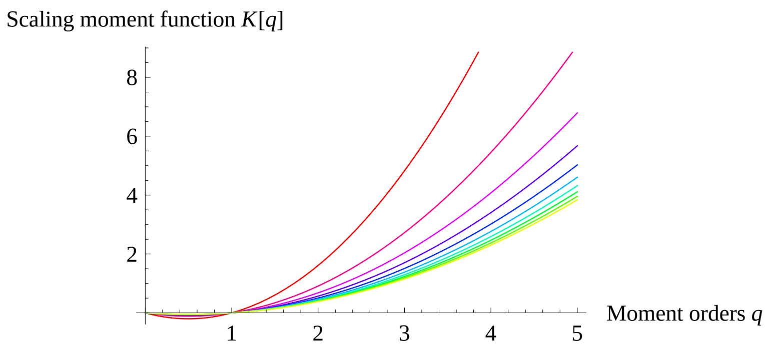

Figure 2: Graphs of the scaling moment function $K(q,k)$ for moment orders $q = 0, 5$, cascade levels $k = 1, 10$ (from top to bottom (for large q's) /from red to yellow) and the set of parameters chosen by the authors.

field;

- the index of multifractality $\alpha = d^2K(q)/dq^2|_{q=1}/C_1$ characterises both the curvature of $K(q)$ at $q = 1$ and how the intermittency relatively varies when departing from the mean field. Its lower bound $\alpha = 0$ corresponds to a uni/mono-fractal field, whereas its upper bound $\alpha = 2$ to the misnamed lognormal model.

These three characteristics are generally only local, i.e. characterise the statistics of a multifractal field only in the neighbourhood of its mean. They are indeed based on $K(q)$ and its two first derivatives at the point $q = 1$ and are therefore defined by moments of (non integer) orders $q \approx 1$. However, these characteristics became global, i.e. for all orders q's, for the 'universal multifractals' (Schertzer and Lovejoy, 1987, 1997). This 3-parameter family of multifractals is not only large, but also stable and attractive; it corresponds to the limit set of a broad generalisation of the central limit theorem in a multiplicative framework. However, this global characterisation needs to take into account the generic phenomena of multifractal phase transitions, i.e. discontinuities of the effective scaling moment function $K^*(q)$ (estimated over a given number of samples), which is the analogue of a thermodynamic potential, at given critical order q^*, the analogue of the inverse of a critical temperature (θ denotes the Heaviside function):

$$K^*(q) = (1 - \theta(q - q^*))K(q) + \theta(q - q^*)(K(q^*) + \gamma^*(q - q^*)) \tag{2}$$

$$K(q) = \frac{C_1}{\alpha - 1}(q^\alpha - q) \tag{3}$$

For a second order transition $\gamma^* = dK(q)/dq|_{q=q^*}$, whereas for a first order phase transition $\gamma^* > dK(q)/dq|_{q=q^*}$. In both cases, γ^* increases with the log of the number of samples, linearly for the latter case. For an infinite number of samples, the first order transition corresponds to a divergence of the theoretical moments. The authors mentioned this possibility, but ignored the second order phase transition that explains in

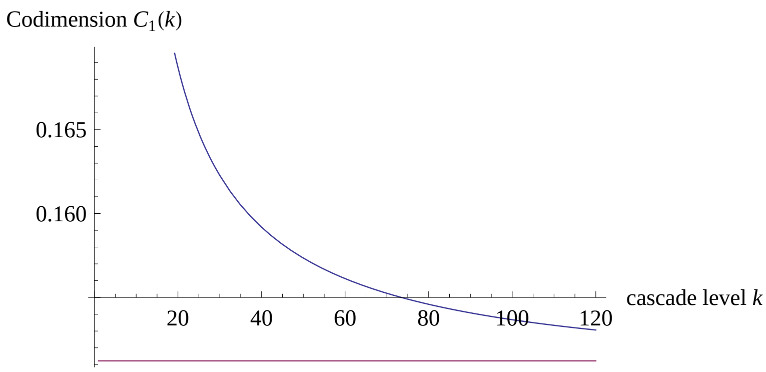

Figure 3: Graph of the codimension $C_1(k)$ of the mean intermittency at the k^{th} level of the cascade ($k = 1, 120$), for the set of parameters chosen by the authors, as well as its as its asymptotic value $C_1(\infty) = 0.15$

a straightforward manner what they numerically observed. Indeed, recalling that in agreement with Eq.1 a multifractal field R has an infinite hierarchy of singularities γ's:

$$R_\lambda \approx \lambda^\gamma \tag{4}$$

The physics of the second order multifractal transition corresponds to the fact that for any finite sample there is a supremum γ_s of the singularities γ's present in this sample. This supremum being an isolated point, its support set has a zero dimension $D(\gamma_s)$, equivalently a codimension $c(\gamma_s) \equiv d - D(\gamma_s) = d$, where d is the dimension of the embedding space ($d = 1$ for a time series). Due to the fact that the codimension $c(\gamma)$ is the Legendre transform of the scaling moment function $K(q)$, and vice-versa, in analogy to the fact that the entropy is the Legendre transform of the thermodynamic potential, one obtains that the critical order q_s corresponding to γ_s is given by:

$$q_s = (d/C_1)^{1/\alpha} \tag{5}$$

To apply these ideas to the present model the only difficulty is to go through the cumbersome algebra of its scale dependent parameters, which implies a rather unusual scale dependence of the scaling moment function $K(q, k)$, see Fig.1. This complexity algebra is first due to the fact that the generator of this discrete cascade is obtained with the help of a discrete fractional gaussian noise that is finally transformed into a log divergent generator with the resolution, as required to obtain a scaling field. To avoid confusion between the fractional integration involved in their cascade generator and a possible fractional integration over the cascade itself, as mentioned above, we denote their corresponding H parameter by h. The differences are further discussed in a companion comment ("Further (mono fractal) limitations of Climactograms", by Lovejoy et al.). In any case, it would have been simpler to directly define a log divergent generator, as usually done. A second complexity is introduced by an attempt to recover a translation invariance in a discrete cascade model. Let us briefly mention, without further discussion in this comment, that the interest and relevance of such an attempt can be brought into question by the fact that discrete cascades are intrinsically built within an ultrametic framework, not a metric one. Furthermore, there already exist well defined continuous (in scale) cascades that are not only translation invariant, but also respect causality, which remains beyond the discrete cascade framework (Marsan et al., 1996, Schertzer et al., 1997). In any case, the presented model has already the drawback that its scaling moment function is not scale invariant, but depends on the level k of the cascade. To simplify the discussion, let us focus on the lognormal case, i.e. $\alpha = 2$ in Eq. 2 with a codimension of the mean intermittency $C_1(k)$ depending on the k^{th} level of the cascade:

$$C_1(k) = Ln[2^{2k(1-h)}(\sigma_0/\mu_0)^2 + 1]] \tag{6}$$

$$C_1(k) \approx 1 - h + 2Log[\sigma_0/\mu_0]/(2kLn2) \tag{7}$$

One may note that the mean intermittency (as measured by $C_1(k)$) increases with scale and therefore brings into question the physical relevance of such a model. Without discussing the (limited) significance of the involved parameters, we use the set of parameter values chosen by the authors ($\mu_0 = 1, \sigma_0 = 1.29, h = .85$) and thus obtain the figures Fig.1 and Fig.2 respectively for $q_s(k)$ and $C_1(k)$, as well as the asymptotic values $C_1(\infty) = .15$ and $q_s(\infty) \approx 2.582$, which are relevant for a "dressed" cascade, i.e. a cascade first downscaled to infinitely small scales, then upscaled back to a finite observation scale.

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