Lecture notes on stochastics –Part 2

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Aspects of stochastics

Entropy production, scaling, climacogram, climacospectrum, generic simulation

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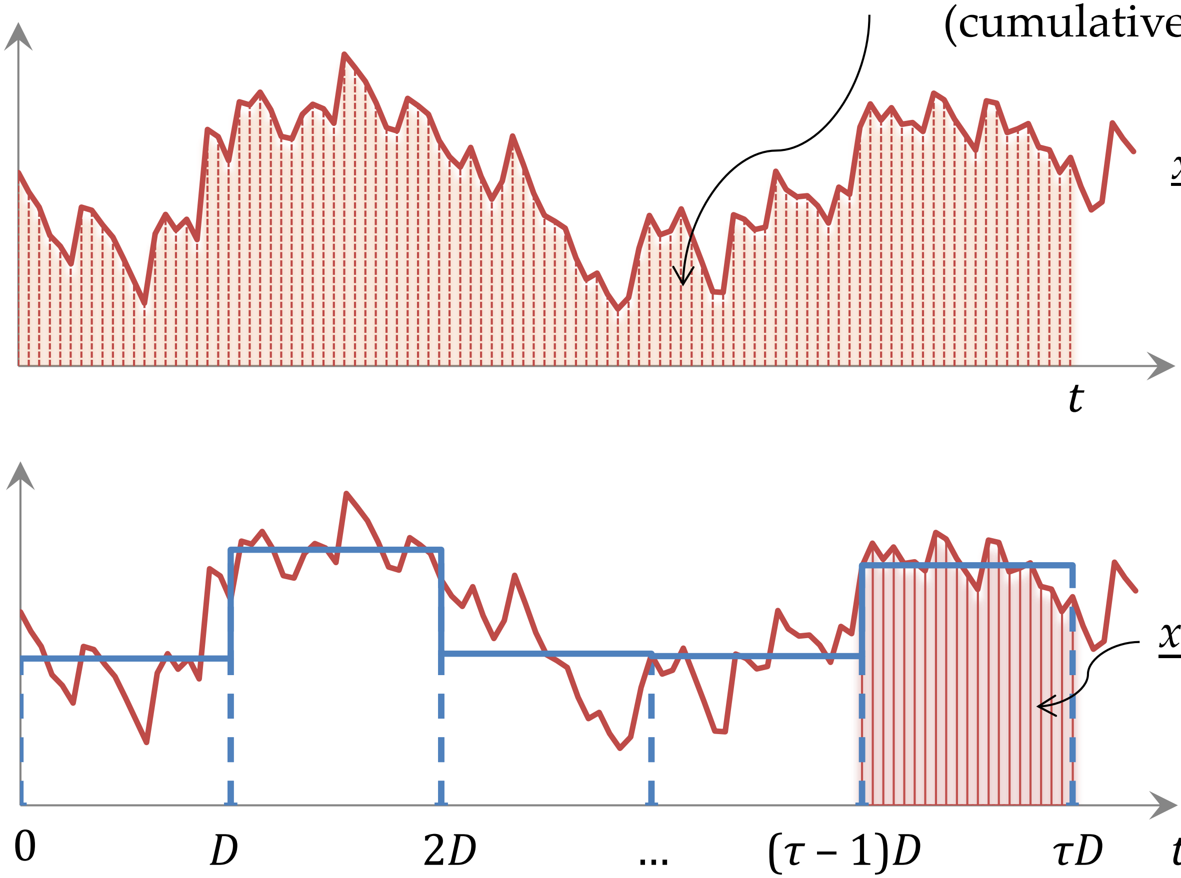
A stochastic process in continuous and discrete time

$\underline{X}(t) := \int_0^t \underline{x}(u)\mathrm{d}u$
(cumulative, nonstationary)

$\underline{x}(t)$ (instantaneous, continuous-time process)



$$\underline{x}_\tau^{(D)} := \frac{1}{D}\int_{(\tau-1)D}^{\tau D} \underline{x}(u)du$$
$$= \frac{1}{D}\left(\underline{X}(\tau D) - \underline{X}((\tau-1)D)\right)$$

(averaged at time scale D)

Note that the graphs display a realization of the process (it is impossible to display the process as such) while the notation is for the process per se.

Definitions and notation – continuous time

Name of quantity or characteristic	Symbol and definition	Remarks	Ref.
Stochastic process of interest	$\underline{x}(t)$	Assumed stationary	
Time, continuous	t	Dimensional quantity	
Cumulative process	$\underline{X}(t) \coloneqq \int_0^t \underline{x}(\xi)\mathrm{d}\xi$	Nonstationary	(1)
Variance, instantaneous	$\gamma_0 \coloneqq \mathrm{Var}[\underline{x}(t)]$	Constant (not a function of t)	(2)
Cumulative climacogram	$\Gamma(t) \coloneqq \mathrm{Var}[\underline{X}(t)]$	A function of t, $\Gamma(0) = 0$	(3)
Climacogram	$\gamma(k) \coloneqq \mathrm{Var}[(1/k)(\underline{X}(t+k) - \underline{X}(t))]$ $= \mathrm{Var}[\underline{X}(k)/k] = \Gamma(k)/k^2$	Not a function of t, $\gamma(0) = \gamma_0$	(4)
Time scale, continuous	k	Units of time	
Autocovariance function	$c(h) := \mathrm{Cov}[\underline{x}(t), \underline{x}(t+h)]$	$c(0) = \gamma_0$	(5)
Time lag, continuous	h	Units of time	
Structure function (or semivariogram or variogram)	$v(h) \coloneqq \frac{1}{2}\mathrm{Var}\left[\underline{x}(t) - \underline{x}(t+h)\right]$		(6)
Climacostructure function	$\xi(k) \coloneqq \gamma_0 - \gamma(k)$		(7)
Power spectrum (or spectral density)	$s(w) \coloneqq 4\int_0^\infty c(h)\cos(2\pi w h)\,\mathrm{d}h$		(8)
Frequency, continuous	$w = 1/k$	Units of inverse time	(9)

Definitions and notation – discrete time

Name of quantity or characteristic	Symbol and definition	Remarks	Ref.
Stochastic process, discrete time	$\underline{x}_\tau^{(D)} := \frac{1}{D}\int_{(\tau-1)D}^{\tau D} \underline{x}(u)\mathrm{d}u = \frac{1}{D}\left(\underline{X}(\tau D) - \underline{X}\big((\tau-1)D\big)\right)$		(10)
Time unit = discretization time step	D	Length of time window of averaging	
Time, discrete	$\tau := t/D$	Dimensionless quantity, integer	(11)
Characteristic variance	$\mathrm{Var}[\underline{x}_\tau^{(D)}] = \gamma(D)$		(12)
Climacogram	$\gamma_\kappa^{(D)} = \gamma(\kappa D) = \frac{\Gamma(\kappa D)}{(\kappa D)^2}$	$\gamma_1^{(D)} = \gamma(D)$	(13)
Time scale, discrete	$\kappa = k/D$	Dimensionless quantity	(14)
Autocovariance function	$c_\eta^{(D)} := \mathrm{Cov}[\underline{x}_\tau^{(D)}, \underline{x}_{\tau+\eta}^{(D)}]$	$c_0^{(D)} = \gamma(D)$	
Time lag, discrete	$\eta = h/D$	Dimensionless quantity	(15)
Structure function	$v_\eta^{(D)} = \gamma(D) - c_\eta^{(D)}$		(16)
Power spectrum	$s_\mathrm{d}^{(D)}(\omega) = \frac{1}{D}\sum_{j=-\infty}^{\infty} s\left(\frac{\omega+j}{D}\right)\,\mathrm{sinc}^2\big(\pi(\omega+j)\big)$		(17)
Frequency, discrete	$\omega = wD = 1/\kappa$	Dimensionless quantity	(18)

Note: In time-related quantities, Latin letters denote dimensional quantities and Greek letters dimensionless ones. The Latin *i, j, l* may also be used as integers to denote quantities τ, η, κ, depending on the context.

Relationships between characteristics of a process in continuous and discrete time

Related characteristics	Symbol and definition	Inverse relationship	Ref.
$\gamma(k) \leftrightarrow c(h)$	$\gamma(k) = 2\int_0^1 (1-\chi)c(\chi k)\mathrm{d}\chi$	$c(h) = \frac{1}{2}\frac{\mathrm{d}^2\left(h^2\gamma(h)\right)}{\mathrm{d}h^2}$	(19)
$s(w) \leftrightarrow c(h)$	$s(w) \coloneqq 4\int_0^\infty c(h)\cos(2\pi wh)\,\mathrm{d}h$	$c(h) = \int_0^\infty s(w)\cos(2\pi wh)\,\mathrm{d}w$	(20)
$\gamma(k) \leftrightarrow s(w)$	$\gamma(k) = \int_0^\infty s(w)\ \mathrm{sinc}^2(\pi wk)\,\mathrm{d}w$	$s(w) \coloneqq 2\int_0^\infty \frac{\mathrm{d}^2\left(h^2\gamma(h)\right)}{\mathrm{d}h^2}\cos(2\pi wh)\,\mathrm{d}h$	(21)
$v(h) \leftrightarrow c(h)$	$v(h) = \gamma_0 - c(h)$	$c(h) = v(\infty) - v(h)\ (v(\infty) = \gamma_0)$	(22)
$\xi(k) \leftrightarrow \gamma(k)$	$\xi(k) \coloneqq \gamma_0 - \gamma(k)$	$\gamma(k) = \xi(\infty) - \xi(k)\ \ (\xi(\infty) = \gamma_0)$	(23)
$\xi(k) \leftrightarrow v(h)$	$\xi(k) = 2\int_0^1 (1-\chi)v(\chi k)\mathrm{d}\chi$	$v(h) = \frac{1}{2}\frac{\mathrm{d}^2\left(h^2\xi(h)\right)}{\mathrm{d}h^2}$	(24)
$\gamma_\kappa^{(D)} \equiv \gamma(\kappa D) \leftrightarrow c_\eta^{(D)}$	$\gamma_\kappa^{(D)} = \frac{1}{\kappa}\left(c_0^{(D)} + 2\sum_{\eta=1}^{\kappa-1}\left(1-\frac{\eta}{\kappa}\right)c_\eta^{(D)}\right)$ Alternatively, $\gamma_\kappa^{(D)} = \frac{\Gamma(\kappa D)}{(\kappa D)^2}$ where, in recursive mode, $\Gamma(\kappa D) =$ $2\Gamma\big((\kappa-1)D\big) - \Gamma\big((\kappa-2)D\big) + 2c_{j-1}^{(D)}D^2$ with $\Gamma(0) = 0$, $\Gamma(D) = c_0^{(D)}D^2$	$c_\eta^{(D)} =$ $\frac{1}{D^2}\left(\frac{\Gamma(\lvert\eta+1\rvert D)+\Gamma\big((\lvert\eta-1\rvert D\big)}{2} - \Gamma(\lvert\eta\rvert D)\right)$	(25)
$c_\eta^{(D)} \leftrightarrow s_\mathrm{d}^{(D)}(\omega)$	$s_\mathrm{d}^{(D)}(\omega) = 2c_0^{(D)} + 4\sum_{\eta=1}^{\infty} c_\eta^{(D)}\cos(2\pi\eta\omega)$	$c_\eta^{(D)} = \int_0^{1/2} s_\mathrm{d}^{(D)}(\omega)\cos(2\pi\omega\eta)\,\mathrm{d}\omega$	(26)
$v_\eta^{(D)} \leftrightarrow c_\eta^{(D)}$	$v_\eta^{(D)} = \gamma(D) - c_\eta^{(D)}$	$c_\eta^{(D)} \coloneqq \gamma(D) - v_\eta^{(D)}$	(27)

Asymptotic power laws and the log-log derivative

It is quite common that functions $f(x)$ defined in $[0, \infty)$, whose limits at 0 and ∞ exist, are associated with asymptotic power laws as $x \to 0$ and ∞ (Koutsoyiannis, 2014b).

Power laws are functions of the form

$$f(x) \propto x^b \tag{28}$$

A power law is visualized in a graph of $f(x)$ plotted in logarithmic axis vs. the logarithm of x, so that the plot forms a straight line with slope b. Formally, the slope b is expressed by the **log-log derivative** (LLD):

$$f^{\#}(x) \coloneqq \frac{\mathrm{d}(\ln f(x))}{\mathrm{d}\,(\ln x)} = \frac{x f'(x)}{f(x)} \tag{29}$$

If the power law holds for the entire domain, then $f^{\#}(x) = b =$ constant. Most often, however, $f^{\#}(x)$ is not constant. Of particular interest are the **asymptotic values** for $x \to 0$ and ∞, symbolically $f^{\#}(0)$ and $f^{\#}(\infty)$, which **define two asymptotic power laws**.

Definition and importance of entropy

Historically entropy was introduced in thermodynamics but later it was given a rigorous definition within probability theory (owing to Boltzmann, Gibbs and Shannon). Thermodynamic and probabilistic entropy are essentially the same thing (Koutsoyiannis, 2013, 2014a; but others have different opinion).

Entropy is a dimensionless measure of uncertainty defined as follows:

For a **discrete random variable** $\underline{z}$ with probability mass function $P_j \coloneqq P\{\underline{z} = z_j\}$

$$\Phi[\underline{z}] := \mathrm{E}[-\ln P(\underline{z})] = -\sum_{j=1}^{w} P_j \ln P_j \qquad (30)$$

For a **continuous random variable** $\underline{z}$ with probability density function $f(z)$:

$$\Phi[\underline{z}] := \mathrm{E}\left[-\ln \frac{f(\underline{z})}{m(\underline{z})}\right] = -\int_{-\infty}^{\infty} \ln \frac{f(z)}{m(z)} f(z) \mathrm{d}z \qquad (31)$$

where $m(z)$ is the density of a background measure (usually $m(z) = 1[z^{-1}]$).

Entropy acquires its importance from the **principle of maximum entropy** (Jaynes, 1957), which postulates that the entropy of a random variable should be at maximum, under some conditions, formulated as constraints, which incorporate the information that is given about this variable.

Its physical counterpart, the tendency of **entropy to become maximal (2nd Law of thermodynamics) is the driving force of natural change**.

Entropy production in stochastic processes

In a stochastic process the change of uncertainty in time can be quantified by the **entropy production,** i.e. the time derivative (Koutsoyiannis, 2011):

$$\Phi'[\underline{X}(t)] := \mathrm{d}\Phi[\underline{X}(t)]/\mathrm{d}t \tag{32}$$

A more convenient (and dimensionless) measure is the **entropy production in logarithmic time** (EPLT):

$$\varphi(t) \equiv \varphi[\underline{X}(t)] := \Phi'[\underline{X}(t)]\, t \equiv \mathrm{d}\Phi[\underline{X}(t)] \,/\, \mathrm{d}(\ln t) \tag{33}$$

For a Gaussian process, the entropy depends on its variance $\Gamma(t)$ only and is given as (cf. Papoulis, 1991):

$$\Phi[\underline{X}(t)] = (1/2)\ln(2\pi \mathrm{e}\, \Gamma(t)/m^2) \tag{34}$$

The EPLT of a Gaussian process is thus easily shown to be:

$$\varphi(t) = \Gamma'(t)\, t \,/\, 2\Gamma(t) = 1 + \gamma'(t)\, t \,/\, 2\gamma(t) = ½\, \Gamma^{\#}(t) = 1 + ½\, \gamma^{\#}(t) \tag{35}$$

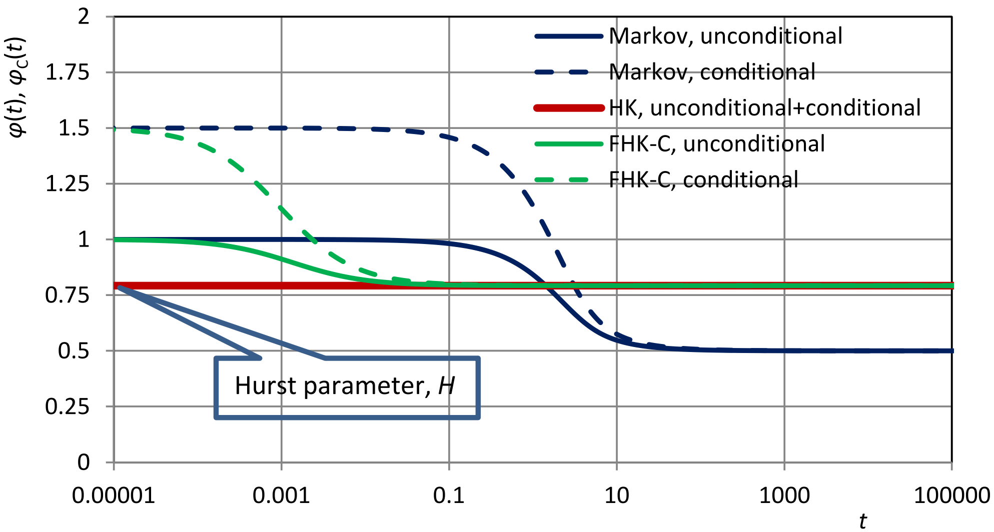
That is, **EPLT** is visualized and estimated by the **slope of a log-log plot of the climacogram.** When the past and the present are observed, instead of the unconditional variance $\gamma(t)$ we should use a variance $\gamma_{\mathrm{C}}(t)$ conditional on the known past and present. This turns out to equal a **differenced climacogram** (Koutsoyiannis, 2017):

$$\gamma_{\mathrm{C}}(k) = \varepsilon\big(\gamma(k) - \gamma(2k)\big), \qquad \varepsilon = \frac{1}{1 - 2^{\gamma^{\#}(\infty)}} \tag{36}$$

The **conditional entropy production in logarithmic time (CEPLT)** becomes:

$$\varphi_C(t) = 1 + ½\gamma_C^{\#}(t) \tag{37}$$

Examples of stochastic processes and their entropy production



All three processes have same:
variance $\gamma(1) = 1$;
autocovariance for lag 1, $c_1^{(1)} = 0.5$;
fractal parameter $M = 0.5$

The HK and FHK processes have Hurst parameter $H = 0.7925$.

Markov process, maximizing entropy production for small times ($t \to 0$) but minimizing it for large times ($t \to \infty$):

$$c(h) = \lambda \mathrm{e}^{-h/\alpha}, \ \gamma(k) = \frac{2\lambda}{k/\alpha}\left(1 - \frac{1-\mathrm{e}^{-k/\alpha}}{k/\alpha}\right) \tag{38}$$

Hurst-Kolmogorov (HK) process, maximizing entropy production for large times ($t \to \infty$) but minimizing it for small times ($t \to 0$):

$$\gamma(k) = \lambda(\alpha/k)^{2-2H} \tag{39}$$

Filtered Hurst-Kolmogorov process with a generalized Cauchy-type climacogram (FHK-C), maximizing entropy production for large ($t \to \infty$) and small times ($t \to 0$):

$$\gamma(k) = \lambda\left(1 + (k/\alpha)^{2M}\right)^{\frac{H-1}{M}} \tag{40}$$

The parameters a and λ are scale parameters. The parameter H is the Hurst parameter and determines the global properties of the process with the notable property $H = \varphi(\infty) = \varphi_C(\infty)$. The parameter M (for Mandelbrot) is the fractal parameter. Both M and H are dimensionless parameters varying in the interval (0, 1] with $M < ½$ or $> ½$ indicating a rough or a smooth process, respectively, and with $H < ½$ or $> ½$ indicating an antipersistent or a persistent process, respectively (see also the graph in p. 12).

The climacospectrum

By slightly modifying the differenced climacogram (in order to make it integrable in $(0, \infty)$), i.e. by multiplying with k, we can obtain an additional tool, which resembles the power spectrum and thus is referred to as the **climacospectrum**:

$$\zeta(k) \coloneqq \frac{k\big(\gamma(k)-\gamma(2k)\big)}{\ln 2} \tag{41}$$

The climacospectrum is also written in an alternative manner in terms of frequency $w = 1/k$:

$$\tilde{\zeta}(w) \coloneqq \zeta(1/w) = \frac{\gamma(1/w)-\gamma(2/w)}{(\ln 2)w} \tag{42}$$

The inverse transformation, i.e., that giving the climacogram $\gamma(k)$ once the climacospectrum $\zeta(k)$ is known, is

$$\gamma(k) = \ln 2 \sum_{i=0}^{\infty} \frac{\zeta(2^i k)}{2^i k} = \gamma(0) - \ln 2 \sum_{i=1}^{\infty} \frac{\zeta(2^{-i} k)}{2^{-i} k} \tag{43}$$

As also happens with the power spectrum, the entire area under the curve $\tilde{\zeta}(w)$ is precisely equal to the variance $\gamma(0)$ of the instantaneous process. The climacospectrum has also the same asymptotic behaviour with the power spectrum, i.e.,

$$\tilde{\zeta}^{\#}(0) = -\zeta^{\#}(\infty) = s^{\#}(0), \qquad \tilde{\zeta}^{\#}(\infty) = -\zeta^{\#}(0) = s^{\#}(\infty) \tag{44}$$

This property holds almost always, with the exception of the cases where $\zeta^{\#}(0)$ is a specific integer ($\zeta^{\#}(\infty) = -1$ or $\zeta^{\#}(0) = 3$).

The climacospectrum is also connected with the CEPLT trough:

$$\varphi_C(k) = ½\left(1 + \zeta^{\#}(k)\right) = ½\left(1 - \tilde{\zeta}^{\#}(1/k)\right) \tag{45}$$

The climacogram and the climacogram-based metrics compared to more standard metrics

- In stochastic processes, almost all classical statistical estimators are biased and uncertain; in processes with LTP bias and uncertainty are very high.
- In the climacogram (variance), bias and uncertainty are easy to control as they can be calculated analytically (and a priori known; see Koutsoyiannis, 2016).
- The autocovariance function is the second derivative of the climacogram.
  - Estimation of the second derivative from data is too uncertain and makes a very rough graph.
  - Estimation of autocovariance is too biased in processes with LTP.
- The power spectrum is the Fourier transform of the autocovariance and entails an even rougher shape and more uncertain estimation than in the autocovariance (see also Dimitriadis and Koutsoyiannis, 2015).
- An additional advantage of the climacogram is its close relationship with entropy production.
- A further advantage is its expandability to high-order moments (see part 3 of the Lecture Notes).

Asymptotic scaling of second order properties

EPLT and the CEPLT are related to LLDs (slopes of log-log plots) of second order tools such as climacogram, climacospectrum, power spectrum, etc. With a few exceptions, these slopes are nonzero asymptotically, hence entailing **asymptotic scaling** or **asymptotic power laws** with the **LLDs being the scaling exponents**. It is intuitive to expect that an emerging asymptotic scaling law would provide a good approximation of the true law for a range of scales.

If the scaling law was appropriate for the entire range of scales, then we would have a simple scaling law. Such simple scaling sounds attractive from a mathematical point of view, but it turns out to be **impossible in physical processes** (Koutsoyiannis, 2017; see also the graph in p. 12).

It is thus physically more realistic to expect **two different types of asymptotic scaling** laws, one in each of the ends of the continuum of scales. The respective scaling exponents are the following:

Local scaling or **smoothness** or **fractal behaviour**, when $k \to 0$ or $w \to \infty$:

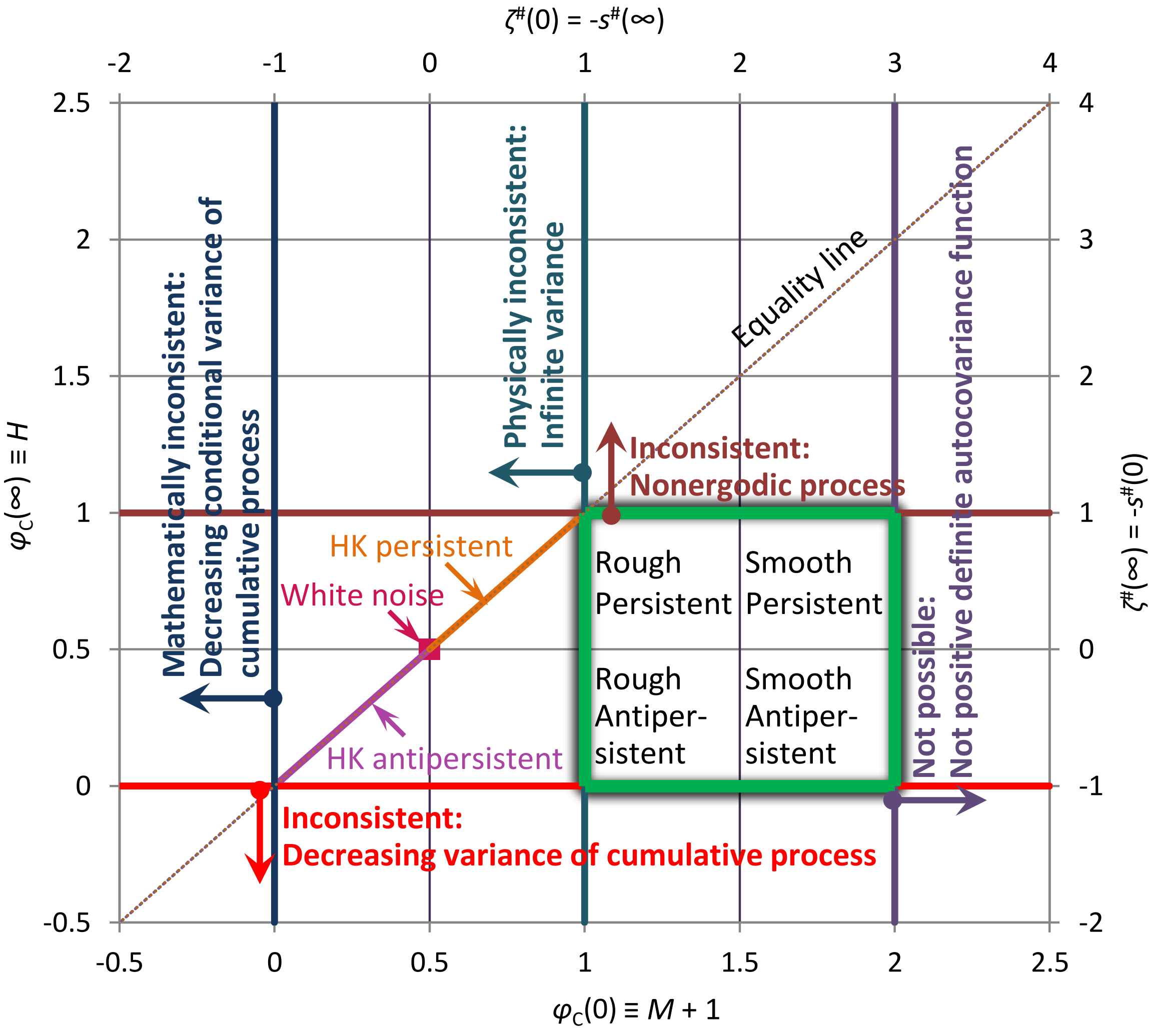
$$\gamma_{\mathrm{C}}^{\#}(0) = \xi^{\#}(0) = v^{\#}(0) = \zeta^{\#}(0) - 1 = 2\varphi_C(0) - 2 = -s^{\#}(\infty) - 1 = 2M \tag{46}$$

Global scaling or **persistence** or **Hurst- Kolmogorov behaviour**, when $k \to \infty$ or $w \to 0$**:**

$$\gamma_{\mathrm{C}}^{\#}(\infty) = \gamma^{\#}(\infty) = c^{\#}(\infty) = \zeta^{\#}(\infty) - 1 = 2\varphi_C(\infty) - 2 = -s^{\#}(0) - 1 = 2H - 2 \tag{47}$$

Here, the emergence of scaling has been related to maximum entropy considerations, and this may provide the theoretical background in modelling complex natural processes by such scaling laws. Generally, scaling laws are a mathematical necessity and could be constructed for virtually any continuous function defined in $(0, \infty)$. In other words, there is no magic in power laws, except that they are, logically and mathematically, a necessity.

Bounds of scaling



Bounds of asymptotic values of CEPLT, $\varphi_C(0)$ and $\varphi_C(\infty)$, and corresponding bounds of the log-log slopes of power spectrum and climacospectrum.
The "green square" represents the admissible region (note that $s^{\#}$ can, by exception, take on values out of the square when $\varphi_C(0) = 2$ or $\varphi_C(\infty) = 0$). The reasons why a process out of the square would be impossible or inconsistent are also marked. The lines $\varphi_C(0) = 3/2$ and $\varphi_C(\infty) = 1/2$ define "neutrality" (which is represented by a Markov process) and support the classification of stochastic processes into the indicated four categories (smaller squares within the "green square").

Stochastic simulation

The so-called symmetric moving average (SMA) method (Koutsoyiannis, 2000) can directly generate time series with any arbitrary autocorrelation function provided that it is mathematically feasible. It consists of the following generation equation which transforms white noise $\underline{v}_i$ averaged in discrete time (and not necessarily Gaussian), to a process $\underline{x}_i$ with the specified autocorrelation:

$$\underline{x}_i = \sum_{l=-q}^{q} a_{|l|} \underline{v}_{i+l} \tag{48}$$

In theory, the limit q should be ∞ but in practice a truncation to a specific finite q is made (see Koutsoyiannis, 2016, for methods to handle the truncation error).

To calculate the series of coefficients a_l we first determine their Fourier transform $s_{\mathrm{d}}^{a}(\omega)$ from the power spectrum of the process, i.e.,

$$s_{\mathrm{d}}^{a}(\omega) = \sqrt{2 s_{\mathrm{d}}(\omega)} \tag{49}$$

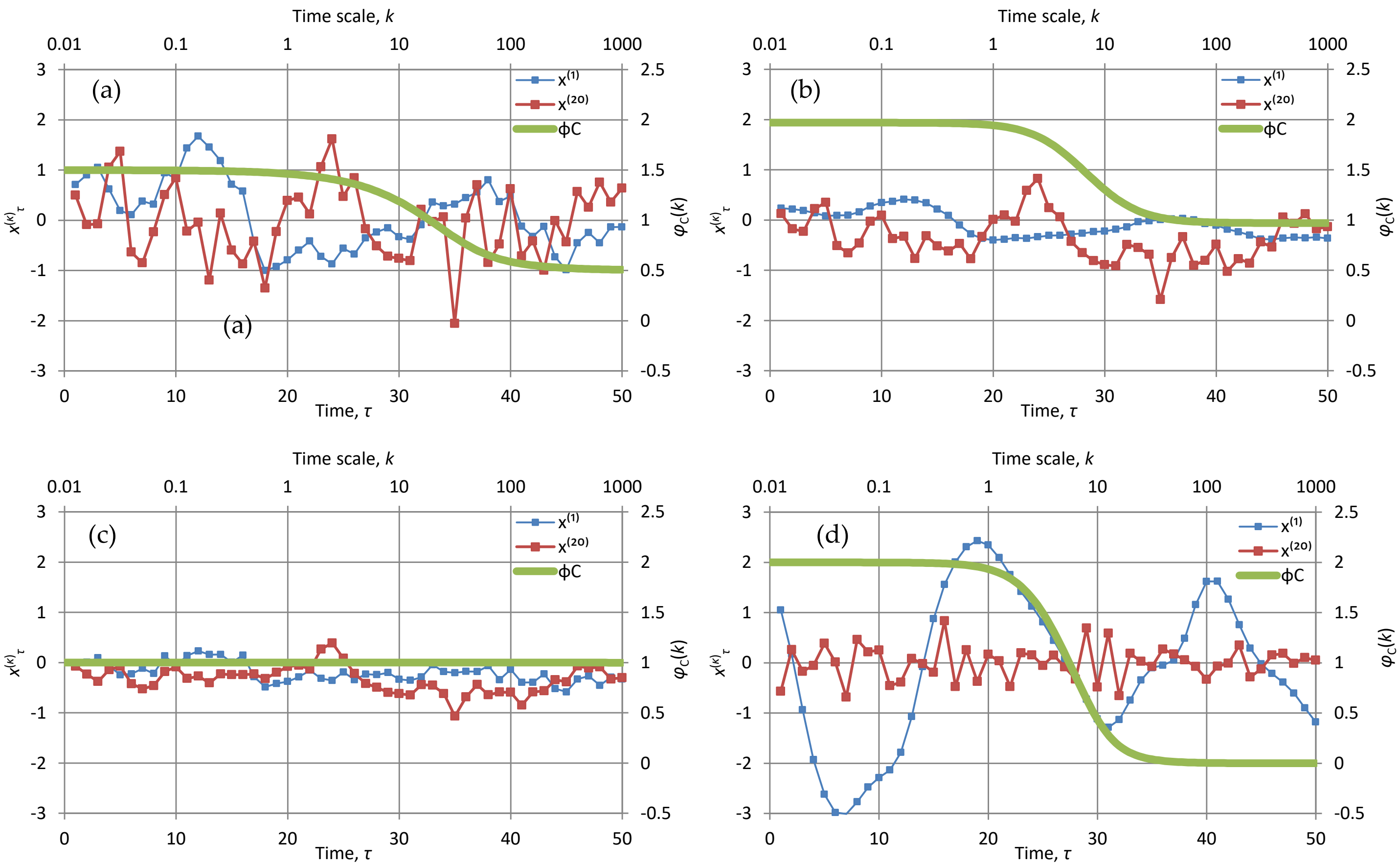
and then we inverse the transform and get the coefficients a_l. Note that the coefficients are internal constants of the model, not model parameters.

For the HK process with $H > 0.5$, there is an explicit analytical solution (Koutsoyiannis, 2016):

$$a_l = \sqrt{\frac{2\Gamma(2H+1)\sin(\pi H)\gamma(\Delta)}{\Gamma^2(H+3/2)\,(1+\sin(\pi H))}} \left(\frac{|l+1|^{H+0.5}+|l-1|^{H+0.5}}{2} - |l|^{H+0.5}\right) \tag{50}$$

By properly calculating the high-order moments of $\underline{v}_i$, we can preserve any moment of $\underline{x}_i$ that we wish (Dimitriadis and Koutsoyiannis, 2018). Thus, the scheme can handle any marginal distribution of $\underline{x}_i$.

Some results of simulations



(a) **Markov**;

(b) FHK, with **CEPLT** close to the **absolute maximum** ($H = M = 0.97$);

(c) FHK, close to "**red noise**", i.e., with CEPLT close to the absolute maximum for large scales ($H = 0.99$) and close to the absolute minimum for small scales ($M = 0.01$);

(d) process with the **blackbody** spectrum, i.e. with CEPLT equal to the absolute minimum (0) for large scales and to the absolute maximum (2) for small scales.

The first fifty terms of times series at time scales $k = 1$ and 20 of time series produced by various models, along with "stamps" of the models (green lines plotted with respect to the secondary axes) represented by the CEPLT, $\varphi_C(k)$. In all cases the discretization time scale is $D = 1$, the characteristic time scale $a = 10$, and the characteristic variance scale λ is chosen so that for time scale D, $\gamma(D) = 1$. The mean is 0 in all cases and the marginal distribution is normal (see details in Koutsoyiannis, 2017).

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