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# APPLICATION OF THE INTEGRATED FINITE DIFFERENCE METHOD IN GROUNDWATER FLOW

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## 1. Abstract

The dominant methods used today for solving partial differential equations (PDE) are the Finite Difference Method (FDM), the Finite Element Method (FEM), the Finite Volume Method (FVM) and the Boundary Element Method (BEM), with FDM and FEM being the most widely used in hydrogeologic modelling. FDM appears to have greater applicability maybe as a result of the simplicity of grid construction and of the solution procedure that it uses. On the other hand, the poor capacity of FDM in representation of complex geometries due to compulsory use of rectangular discretisation makes in some cases inevitable the application of FEM or BEM. In cases where computational time is critical, the so called Integrated Finite Difference Method (IFDM) (Narasimhan and Witherspoon, 1976) that is a variant of the FVM may be a better candidate. This method can be applied successfully with non rectangular discretisation with a small number of cells following the concept of groundwater multicell models (Bear, 1979). The theoretical basis of IFDM along with two applications, which demonstrate that reliable solutions can be achieved even with a very sparse discretisation, are presented.

## 2. FVM

The ground water flow equation is:

$$\mathrm{div}\,(\mathbf{K}\ \mathrm{grad}\ h) + G = \mathrm{SS}\,\frac{\partial h}{\partial t} \qquad (2.1)$$

Integrating this equation and using the divergence theorem, it is obtained:

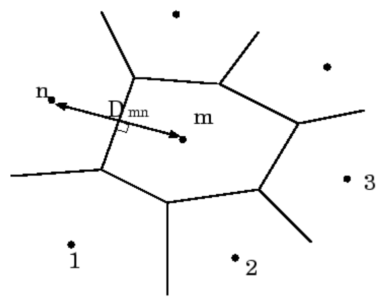
$$\int_S \mathbf{K}\ \mathrm{grad}\ h \cdot \mathbf{n}\ \mathrm{d}S + G\,V = \mathrm{SS}\ V\frac{\partial h}{\partial t} \qquad (2.2)$$

In the above equations, $h$ is the hydraulic head [L], $\mathbf{K}$ is the tensor of hydraulic conductivity [$LT^{-1}$], $\mathbf{n}$ is the unit vector perpendicular to the surface $S$ [$L^2$] that surrounds the volume $V$ [$L^3$] of a cell, $G$ [$T^{-1}$] is the volumetric flux per unit volume and SS [$L^{-1}$] is the specific storage.

The surface integral on the left side of (2.2) is calculated using a numerical method like Gaussian quadrature (Moroney and Turner, 2004). This results in a set of linear equations with unknowns the heads $h$ at the discretisation cells.

## 3. IFDM- Multicell Models

If the edges of the discretisation cells lie exclusively either parallel to no-flow lines or parallel to equipotential lines, then the equation (2.2) can be greatly simplified. In the case of cell $m$ surrounded by $N$ cells the equation (2.2) for cell $m$ is written as:




$$G_m\,V_m + \sum_n K_{mn}\frac{h_n - h_m}{D_{mn}}A_{mn} = \mathrm{SS}_m\,V_m\frac{\Delta h_m}{\Delta t} \qquad (3.1)$$

where $A_{mn}$ is the area of interface between cells $m$ and $n$, and $D_{mn}$ is the distance of the centers of cells $m$ and $n$. The (grad $h$) in equation (2.2) is approximated with $(h_n - h_m)/D_{mn}$. This approximation is accurate in the cases where the common edge of cells $m$ and $n$ is perpendicular to the line that connects the centers of the cells.

## 4. Comparison of methods

The pros and cons of FEM, FDM (Pruist et al., 1993) and IFDM are:

- FEM: Local refinement of grid (adaptive mesh generation) due to non rectangular grids, good accuracy, stability, representation of the spatial variation of anisotropy; on the other hand computationally consuming and relative cumbersome in application.

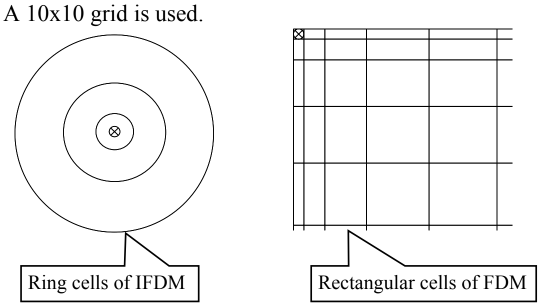
- FDM: Simplicity of theory and algorithm, easiness of application; on the other hand inefficient refinement of grid and poor geometry representation due to the strict use of rectangular grids; also, no standard method to implement the Neumann boundary condition (often trickily implemented with wells).

- IFDM: Same advantages as FDM plus the ability to use non rectangular cells; same disadvantages but with the need of grid geometry to satisfy the two conditions (cell edges either parallel to no-flow lines or parallel to equipotential lines and common edge of cells perpendicular to the line that connects their centers).

## 5. Application 1

In this application an injection well in an infinite homogeneous aquifer is modelled. The problem is solved using FDM (MODFLOW), and IFDM, and the results are compared with the analytical solution (Theis equation).

The problem is one dimensional when using cylindrical coordinates. For this reason, 10 rings centered at the borehole are used in IFDM. Due to symmetry, in FDM the problem can be solved into one quartile of domain but it remains 2D. A 10x10 grid is used.




In the case of isotropic aquifer, IFDM gave more accurate results (see Table) and was much faster (10 instead of 10x10 cells) than FDM.

| Distance (m) | Theis | IFDM | FDM |
|---|---|---|---|
| 0.55 | 121.5 | 121.6 | 122.9 |
| 8.5 | 112.8 | 112.8 | 114.4 |
| 137 | 104.0 | 104.0 | 104.8 |

Hydraulic heads (m) calculated with the three methods at various distances from the borehole

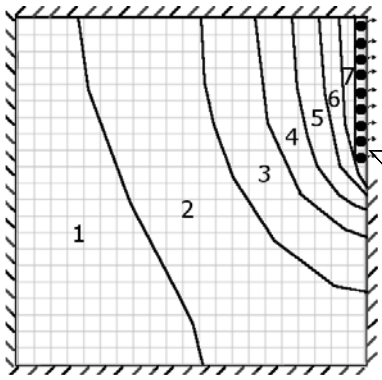
In the case of anisotropic aquifer, the domain is transformed using the formula (Strack, 1999):

$$X = x^* \qquad Y = \sqrt{K_1 / K_2}\; y^* \tag{5.1}$$

to the equivalent isotropic aquifer with conductivity $K = \sqrt{K_1 K_2}$. In this case, IFDM gave also better results compared to FDM especially in the vicinity of the borehole.

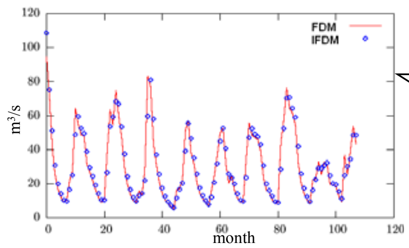
## 6. Application 2

A hypothetical rectangular 30000 m x 30000 m aquifer which discharges to a series of springs on the upper right side and is recharged from an infiltration time series (calculated from a real watershed) is solved with FDM (MODFLOW) and with IFDM.




Number of cells
FDM: 21x21=441
IFDM: 7

Ratio of computational time FDM/IFDM=60/1




The two methods gave virtually the same results in terms of spring discharge.

In terms of time series of hydraulic heads the determination coefficient expressing the difference of the two models varies from 0.85 to 0.95* (FDM cells are grouped according to the corresponding IFDM discretisation in 7 groups, the heads of cells in each group are aggregated resulting in 7 time series).

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* The solution of FDM is considered more reliable because the two conditions of IFDM are not fully satisfied.

## References


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## 7. Conclusions

- Multicell models are mathematically equivalent with IFDM (variant of FDM) if cell edges are either parallel to no-flow lines or parallel to equipotential lines (1st condition) and if the common edges of cells are perpendicular to the line that connects their centers (2nd condition).
- IFDM is much faster than the common numerical methods, which makes this method advantageous in the applications where computational time is critical while there is no need for detailed hydraulic head information.