4. An entropic approach to rainfall – Step 2
• The constant mean constraint in rainfall modeling does not result from a natural principle – as for instance in the physics of an ideal gas, where it represents the availability of one degree of freedom.
• Although it is reasonable to assume a specific mean rainfall, we can allow this to vary in time.
• In this case we can assume that the mean at time \( t \) is the realization of a random process \( M(t) \) with mean \( \mu \) and lag 1 autocorrelation \( \rho = \rho(t) \).
• Application of the ME principle will produce that \( M(t) \) is Markovian with exponential distribution.
• Then application of conditional distribution algebra results in
\[ f(x|t) = \frac{1}{\sqrt{2\pi} \rho(t)} \exp\left(-\frac{(x - \mu)^2}{2\rho^2(t)}\right) \]
• The moments of this distribution are \( \langle x \rangle = \mu \) and \( \langle x^2 \rangle = \mu^2 + 2\rho(t) \mu \).
• The dependence structure becomes more complex than Markovian (difficult to find an analytical solution).

5. An entropic approach to rainfall – Step 3
• Proceeding in a similar manner as in step 2, we can now replace the constant mean \( \mu \) of the process \( M(t) \) with a varying mean, represented by another stochastic process \( N(t) \), with mean \( \mu \) and lag 1 autocorrelation \( \rho = \rho(t) \).
• In this manner we can construct a chain of processes, each member of which represents the mean of the previous process.
• By construction, the lag 1 autocorrelations of these processes form a monotonically increasing sequence, i.e. \( \rho(t) > 1 \).
• The scale or change of fluctuation of each of the processes of the chain is a monotonically increasing sequence, i.e. \( \rho(t) > 1 \).
• The (un)conditional mean of all processes is the same, \( \mu \).
• All moments except the first form an increasing sequence as we proceed down the chain; higher moments increase more.
• Analytical handling of the marginal distribution and the dependence structure is very difficult.
• However we can inspect the idea using Monte Carlo simulation.

6. A demonstration using a chain with 3 processes
• Let \( X \), the rainfall rate at time \( t \), be discretized at a fine time scale (e.g. \( 1 \) minute).
• What we definitively know about \( X \), is that \( X = 0 \).
• Maximum of entropy with only this condition is not possible.
• Now let us assume that rainfall has a specified mean \( \mu \).
• Maximum of entropy with constraint \( \mu \) is
\[ H(X) = -\int f(x) \log f(x) \, dx = -\mu \log \mu - (1 - \mu) \log (1 - \mu) \]
results in the exponential distribution: \( f(x) = \mu^x e^{-\mu x} \).
• In addition, let us assume that there is some time dependence of \( X \), quantified by \( X_1, X_2, \ldots, X_n \); this will introduce an additional constraint for the multivariate distribution
\[ H(X_1, X_2, \ldots, X_n) = -\sum_{i=1}^n \int f_{X_i}(x_i) \log f_{X_i}(x_i) \, dx_i \]
• Here \( \rho \) is the correlation coefficient (\( 0 < \rho < 1 \)) and \( \sigma \) is the standard deviation (for the exponential distribution \( \mu = \sigma \) and thus \( \rho = \sigma^2 / \mu^2 = 1 / \mu \)).
• Entropy maximization in multivariate setting will result in Markovian dependence.

7. Simulation results – distribution function
• Logarithmic plot of “rainfall intensity” \( i \) vs. empirically estimated return period \( T \) (where \( f(i) \) is the distribution function).
• As the number of processes in the chain increases, the right tail of the distribution moves toward higher “rainfall intensity” values and its shape changes from exponential type to power type; simultaneously the probability density becomes infinite for \( x = 0 \).

8. Simulation results – dependence structure
• Logarithmic plot of autocorrelation coefficient \( \rho \) (vs. lag).
• As the number of processes in the chain increases, the shape of the autocorrelation function changes from Markovian (exponential decay – short range dependence) to power type (long range dependence).
• The latter type is characteristic of the Hurst-Kolmogorov behaviour, which can be represented by a simple scaling stochastic process (SSP).

9. Simulation results – variation of the aggregated process
• Logarithmic plot of standard deviation \( \sigma \) of the process aggregated at scale \( k \), vs. scale \( k \).
• The slope of the logarithmic plot (as \( k \to \infty \)) is \( H - 1 \) where \( H \) is the Hurst exponent.
• The slope in the “1 EM” case is \(-0.5\), i.e. \( H = 0.5 \), meaning no Hurst-Kolmogorov behaviour.
• The slope in “3 EM” is \(-0.2\), i.e. \( H = 0.8 \), suggesting a Hurst-Kolmogorov behaviour.

12. Conclusions
• The principle of maximum entropy provides a sound theoretical basis for studying natural phenomena.
• However, maximization of the classical Boltzmann-Gibbs-Shannon entropy at a specific time scale, assuming a specified mean, would result in an exponentially distributed Markovian process, which is unrealistic.
• This can be remedied by performing the entropy maximization in several steps, thus representing rainfall as a chain of stochastic processes, each member of which represents the mean of the previous process and has lag one autocorrelation greater than that of the previous process.
• Monte Carlo simulation demonstrates that such a chain with only these constraints is quite realistic.
• The resulting model is characterized by high autocorrelation at fine scales, close to the one-dimensional (i.e. Hurst-Kolmogorov behaviour), by long distribution tails, and by probability density tending to infinity for rainfall intensity tending to zero.

References