Bilinear surfaces smoothing for spatial interpolation with optional incorporation of an explanatory variable. Part 2: Application to synthesized and rainfall data

Nikolaos Malamos[1*] and Demetris Koutsoyiannis[2]

[1]*Department of Agricultural Technology, Technological Educational Institute of Western Greece, Amaliada, Greece*
[2]*Department of Water Resources and Environmental Engineering, School of Civil Engineering, National Technical University of Athens, Zographou, Greece*
*nmalamos@teimes.gr

Abstract The non-parametric mathematical framework of Bilinear Surface Smoothing (BSS) methodology provides flexible means for spatial (two dimensional) interpolation of variables. As presented in a companion paper, interpolation is accomplished by means of fitting consecutive bilinear surfaces into a regression model with known break points and adjustable smoothing terms defined by means of angles formed by those bilinear surfaces. Additionally, the second version of the methodology (BSSE) incorporates, in an objective manner, the influence of an explanatory variable available at a considerable denser dataset. In the present study, both versions are explored and illustrated using both synthesized and real world (hydrological) data, and practical aspects of their application are discussed. Also, comparison and validation against the results of commonly used spatial interpolation methods (Inverse Distance Weighted, Spline, Ordinary Kriging and Ordinary Cokriging) is performed in the context of the real world application. In every case, the method's efficiency to perform interpolation between data points that are interrelated in a complicated manner was confirmed. Especially during the validation procedure presented in the real world case study, BSSE yielded very good results, outperforming those of the other interpolation methods. Given the simplicity of the approach, the proposed mathematical framework overall performance is quite satisfactory, indicating its applicability for diverse tasks of scientific and engineering hydrology and beyond.

Key words bilinear surface smoothing; spatial interpolation methods; explanatory variable; generalized cross-validation (GCV); rainfall

INTRODUCTION

With the increasing number of applications for environmental purposes, there is also a growing concern about spatially distributed estimates of environmental variables. Analysis and simulation models prior to their application, require tasks such as interpolation between measurements, prediction, filling in missing values in time series, estimation and removal of the measurement errors, etc.

However, most data for environmental variables (soil properties, weather) are collected from point sources. The spatial array of these data may enable a more precise estimation of the value of properties at unsampled sites than simple averaging between sampled points. The value of a property between data points can be interpolated by fitting a suitable model to account for the expected variation (Hartkamp *et al.* 1999). Currently, a lot of methods exist which can accomplish those tasks using appropriate computer codes. They fall into three categories (Li and Heap 2008):

(1) Non-geostatistical methods such as: Splines, Thin Plate Splines (Craven and Wahba 1979, Wahba and Wendelberger 1980) and Regression Methods (Davis 1986)
(2) Geostatistical methods including different approaches of Kriging, such as: Ordinary and Universal Kriging, Kriging with an External Drift or Cokriging (Goovaerts 1997, Burrough and McDonnell 1998) and

(3) Combined methods such as: Trend Surface Analysis Combined with Kriging (Wang *et al*. 2005) and Regression Kriging (Hengl *et al*. 2007).

In the present study applications of an innovative concept are demonstrated. The main idea, presented as Bilinear Surface Smoothing (BSS), is to approximate a surface that may be drawn for the data points (x_i, y_i) with consecutive bilinear surfaces which can be numerically estimated by means of a least squares fitting procedure into a surface regression model with known break points and adjustable weights defined by means of angles formed by those bilinear surfaces. Based on this concept, the second version of the methodology (BSSE) focuses in the combination of two bilinear surfaces into the surface regression model. The first surface is fitted to the available data points while the second incorporates, in an objective manner, the influence of an explanatory variable available at a considerable denser dataset.

Both versions are illustrated using two approaches: (a) theoretical exploration, and (b) real world application in spatial interpolation of rainfall data using the surface elevation as explanatory variable where applicable. In the context of the second application, a comparison with the results of commonly used methodologies like: Inverse Distance Weighted, Spline, Ordinary Kriging and Ordinary Cokriging is performed (Goovaerts 1997, Goovaerts 2000, Burrough and McDonnell 1998). This comparison is implemented twice, firstly by using the entire dataset as input data and secondly, for validation purposes, the original dataset is divided in two subsets: one acts as input dataset while the second subset, that contains the remaining stations, is the validation dataset.

THEORY AND DEFINITIONS

The proposed mathematical framework suggests that fit is meant in terms of minimizing the total square error among the set of original points $z_i(x_i, y_i)$ for $i = 1,\ldots, n$ and the fitted bilinear surface, that in matrix form, can be written as:

$$p = \|\mathbf{z} - \hat{\mathbf{z}}\|^2 \tag{1}$$

where $\mathbf{z} = [z_1,\ldots, z_n]^\mathrm{T}$ is the vector of known applicates of the given data points with size n (the superscript T denotes the transpose of a matrix or vector) and $\hat{\mathbf{z}} = [\hat{z}_1,\ldots,\hat{z}_n]^\mathrm{T}$ is the vector of estimates with size n.

A brief presentation of the method and its equations follows, while the details of the method including the algorithms and derivations of the equations are found in the companion paper (Malamos and Koutsoyiannis submitted). Let $(\mathrm{cx}_l, \mathrm{cy}_k)$, $l = 0, \ldots, \mathrm{mx}$, $k = 0, \ldots, \mathrm{my}$, be a grid of $(\mathrm{mx}+1) \times (\mathrm{my}+1)$ points on the xy plane, so that the rectangle with vertices $(\mathrm{cx}_0, \mathrm{cy}_0)$, $(\mathrm{cx}_{\mathrm{mx}}, \mathrm{cy}_0)$, $(\mathrm{cx}_0, \mathrm{cy}_{\mathrm{my}})$ and $(\mathrm{cx}_{\mathrm{mx}}, \mathrm{cy}_{\mathrm{my}})$ contain all (x_i, y_i). For simplicity we assume that the points on both axes are equidistant, i.e. $\mathrm{cx}_l - \mathrm{cx}_{l-1} = \delta_x$ and $\mathrm{cy}_k - \mathrm{cy}_{k-1} = \delta_y$.

The general estimation function for point u on the $(x\ y)$ plane, according to the Bilinear Smoothing Surface (BSS), method is:

$$\hat{z}_u = d_u \tag{2}$$

while according to the Bilinear Smoothing Surface with explanatory variable (BSSE) is:

$$\hat{z}_u = d_u + t_u\, e_u \tag{3}$$

where, d_u, e_u are the values of the two bilinear surfaces at that point and t_u is the corresponding value of the explanatory variable.

The above equations can be more concisely written, for all given points $z_i(x_i, y_i)$ simultaneously, in the form:

$$\hat{\boldsymbol{z}} = \boldsymbol{\Pi}\, \boldsymbol{d} \tag{4}$$

and

$$\hat{\boldsymbol{z}} = \boldsymbol{\Pi}\, \boldsymbol{d} + \boldsymbol{T}\, \boldsymbol{\Pi}\, \boldsymbol{e} \tag{5}$$

where $\boldsymbol{d} = [d_0,\ldots,d_m]^{\mathrm{T}}$ is a vector of unknown applicates of the bilinear surface d, with size m+1 (m = (mx + 1) × (my + 1) – 1); $\boldsymbol{e} = [e_0,\ldots,e_m]^{\mathrm{T}}$ is a vector of unknown applicates of the bilinear surface e, with size m+1; $\boldsymbol{T}$ is a $n\times n$ diagonal matrix with elements:

$$\boldsymbol{T} = \mathrm{diag}(t(x_1, y_1),\, \ldots,\, t(x_n, y_n)) \tag{6}$$

with $t(x_1, y_1)$, …, $t(x_\mathrm{n}, y_n)$ being the values of the explanatory variable at the given data points; and $\boldsymbol{\Pi}$ is a matrix with size $n\times(m+1)$, whose ijth entry (for i=1, …,n; j=0, …m) is:

$$\pi_{ij} = \begin{cases} \dfrac{(\mathrm{cx}_l - x_i)\,(\mathrm{cy}_k - y_i)}{\delta_x\, \delta_y}, & \text{when } \mathrm{cx}_{l-1} < x_i \le \mathrm{cx}_l \text{ and } \mathrm{cy}_{k-1} < y_i \le \mathrm{cy}_k \\ \dfrac{(\mathrm{cx}_l - x_i)\,(y_i - \mathrm{cy}_{k-1})}{\delta_x\, \delta_y}, & \text{when } \mathrm{cx}_{l-1} < x_i \le \mathrm{cx}_l \text{ and } \mathrm{cy}_k \le y_i < \mathrm{cy}_{k+1} \\ \dfrac{(x_i - \mathrm{cx}_{l-1})\,(y_i - \mathrm{cy}_{k-1})}{\delta_x\, \delta_y}, & \text{when } \mathrm{cx}_l \le x_i < \mathrm{cx}_{l+1} \text{ and } \mathrm{cy}_k \le y_i < \mathrm{cy}_{k+1} \\ \dfrac{(x_i - \mathrm{cx}_{l-1})\,(\mathrm{cy}_k - y_i)}{\delta_x\, \delta_y}, & \text{when } \mathrm{cx}_l \le x_i < \mathrm{cx}_{l+1} \text{ and } \mathrm{cy}_{k-1} < y_i \le \mathrm{cy}_k \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

The calculation of the unknown vectors $\boldsymbol{d}$ and $\boldsymbol{e}$ requires also the definition of matrices $\boldsymbol{\Psi}_x$ and $\boldsymbol{\Psi}_y$ with size $(m-1)\times(m+1)$ (for i=1, …,m–1 and j=0, …m) and ijth entry:

$$\psi_{x\, i,j} = \begin{cases} 2, & \text{when } i=j \quad \text{and } i-k(\mathrm{mx}+1) \notin \{1, \mathrm{mx}+1\} \\ -1, & \text{when } |i-j|=1 \text{ and } i-k(\mathrm{mx}+1) \notin \{1, \mathrm{mx}+1\} \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

where k = 0, …, my, while:

$$\psi_{y\, i,j} = \begin{cases} 2, & \text{when } i=j \quad \text{and } i-l(\mathrm{my}+1) \notin \{1, \mathrm{my}+1\} \\ -1, & \text{when } |i-j|=1 \text{ and } i-l(\mathrm{my}+1) \notin \{1, \mathrm{my}+1\} \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

with l = 0, …, mx (note that $\boldsymbol{\Psi}_x$ and $\boldsymbol{\Psi}_y$ are identical when mx = my).

In the case of BSS the solution that minimizes error, has the following form:

$$\boldsymbol{d} = (\boldsymbol{\Pi}^{\mathrm{T}}\, \boldsymbol{\Pi} + \lambda_x\, \boldsymbol{\Psi}_x^{\mathrm{T}}\, \boldsymbol{\Psi}_x + \lambda_y\, \boldsymbol{\Psi}_y^{\mathrm{T}}\, \boldsymbol{\Psi}_y)^{-1}\, (\boldsymbol{\Pi}^{\mathrm{T}} \boldsymbol{z}) \tag{10}$$

Likewise, in the case of BSSE the solution is:

$$\begin{bmatrix} \boldsymbol{d} \\ \boldsymbol{e} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{\Pi} + \lambda_x \boldsymbol{\Psi}_x{}^{\mathrm{T}}\boldsymbol{\Psi}_x + \lambda_y \boldsymbol{\Psi}_y{}^{\mathrm{T}}\boldsymbol{\Psi}_y & \boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Pi} \\ \boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Pi} & \boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{T}\boldsymbol{\Pi} + \mu_x \boldsymbol{\Psi}_x{}^{\mathrm{T}}\boldsymbol{\Psi}_x + \mu_y \boldsymbol{\Psi}_y{}^{\mathrm{T}}\boldsymbol{\Psi}_y \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{z} \\ \boldsymbol{\Pi}^{\mathrm{T}}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{z} \end{bmatrix} \quad (11)$$

The minimum number of $m + 1$ points required to solve equation (10) or (11) is 6, since the minimum number points needed to define the bilinear surfaces, is the number of points that define two consecutive planes oriented according to either x or y direction. Based on the above equations, we can estimate the applicate of any point that lies in the two-dimensional interval ($[cx_0, cx_{mx}] \times [cy_0, cy_{my}]$) by using either one versions of the proposed methodology.

CHOICE OF PARAMETERS

The adjustable parameters required to implement each of the two versions of the methodology, can be estimated by transforming the smoothing parameters λ and μ in terms of tension: τ_λ and τ_μ, whose values are restricted in the interval [0, 1), for both directions (Malamos and Koutsoyiannis submitted). This transformation provides a convenient search in terms of computational time and is based on the generalized cross-validation (GCV - Craven and Wahba 1979, Wahba and Wendelberger 1980) methodology and symmetric linear smoothers (Buja *et al*. 1989; Carmack *et al*. 2012). Thus, for a given combination of segments mx, my, the minimization of GCV, results in the optimal values of $\tau_{\lambda x}$, $\tau_{\lambda y}$ and $\tau_{\mu x}$, $\tau_{\mu y}$. This can be repeated for several trial combinations of mx, my values, until the global minimum of GCV is reached.

RESULTS AND COMMENTS

We present two applications, the first being synthesized for exploration purposes while the second corresponds to a real world problem, namely spatial interpolation of a rainfall field.

Exploration application

The first application is the implementation of the above presented versions of the methodology, namely BSS and BSSE, in interpolation - fitting to random data points obtained from the generating function (Fig. 1):

$$z(x, y) = (x + 2y - t)^2 + (2x + y - t)^2 + \varepsilon \quad (12)$$

where ε is an represents an intentionally added lognormal error with mean of logarithms 0 and standard deviation of logarithms 0.05.

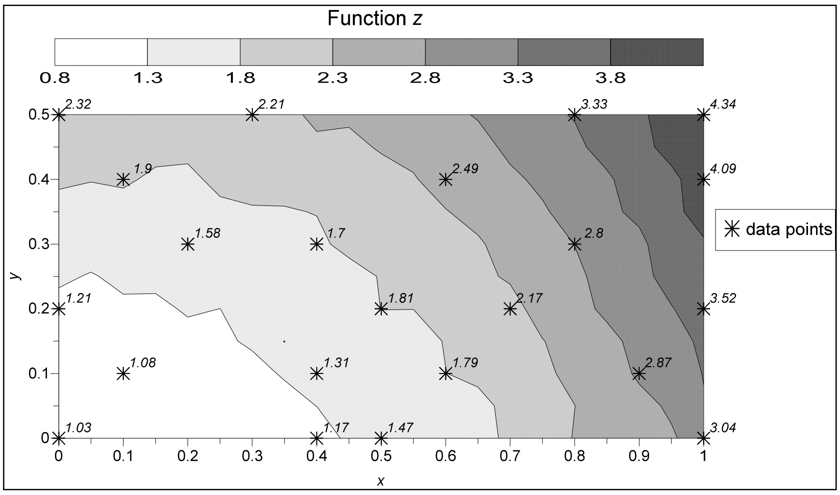


Fig. 1 Generating function, z, along with the twenty-two data points used for the purposes of the exploration application.

Variable t depends on both x and y and acts as the explanatory variable in the case of interpolation with BSSE (Fig. 2):

$$t(x, y) = x\, e^{(y - 0.5x)} \tag{13}$$

The main objective of this application, apart from illustrating the proposed methodology performance, is the investigation of the adjustable parameters variation and the confirmation that the proposed technique for acquiring the global minimum through the generalized cross-validation (GCV) satisfies the method's requirements.

In order to achieve this, we implemented both versions for different numbers of segments mx and my ($1 \leq \text{mx} \leq 15$ and $1 \leq \text{my} \leq 15$, while $m + 1 \geq 6$) using 22 data points ($i = 22$), derived from equation (12). The size of the analysis grid was selected to be 0.05 for $0 \leq x \leq 1$ and $0 \leq y \leq 0.5$, resulting to a total of 231 points at which the generating function was estimated.

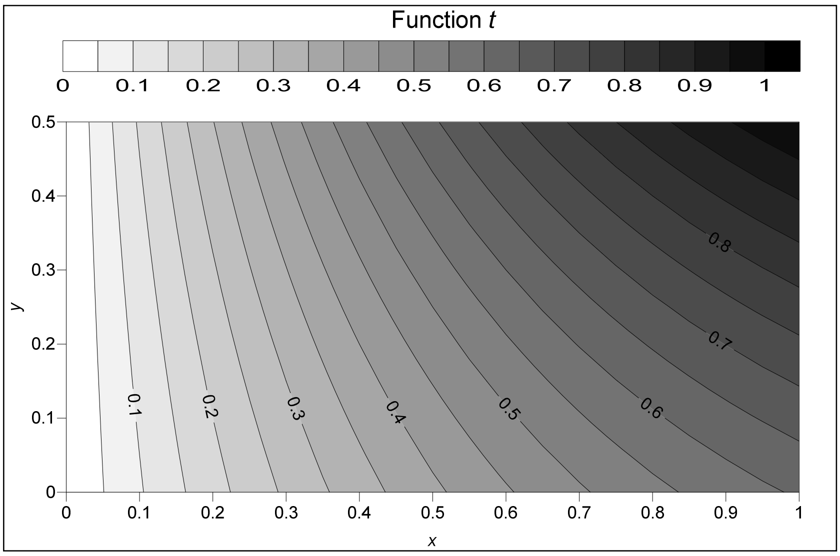


Fig. 2 Explanatory function $t(x, y)$ for the purposes of the exploration application.

Interpolation using Bilinear Surfaces Smoothing (BBS)

After the implementation of the iterative procedure for acquiring the global minimum of GCV as described previously, we obtained the optimal values of the four adjustable parameters: the number of intervals, mx, my, and the smoothing parameters $\tau_{\lambda x}$ and $\tau_{\lambda y}$, as presented in Table 1:

Table 1 BSS parameters optimal values and performance indices, for the exploration application.

number of segments, mx	number of segments, my	$\tau_{\lambda x}$	$\tau_{\lambda y}$	MSE	Global minimum GCV
3	2	0.001	0.002	1.63×10^{-3}	7.87×10^{-3}

The optimal value of the smoothing parameter $\tau_{\lambda x}$ was the minimum allowed value used during the minimization procedure, suggesting that the optimal solution of the problem required that the difference of slopes between the consecutive segments of the bilinear surface according to *x* direction should be as small as possible. The use of a smaller number as lower limit for the smoothing parameters, did not significantly improve the results so for practical reasons the minimum value for the smoothing parameters was set to 0.001.

Figure 3 presents the bilinear surface *d* acquired from the solution of equation (10), along *x* and *y* axes, by using the above presented parameters. The open circles represent the values of vector ***d***, while the available data points are indicated with stars. The consistency to the mathematical framework is verified by the obvious difference of slopes between the consecutive segments of the bilinear surface according to both directions and also the fact that at least one data point is included in each one of the formed rectangles.

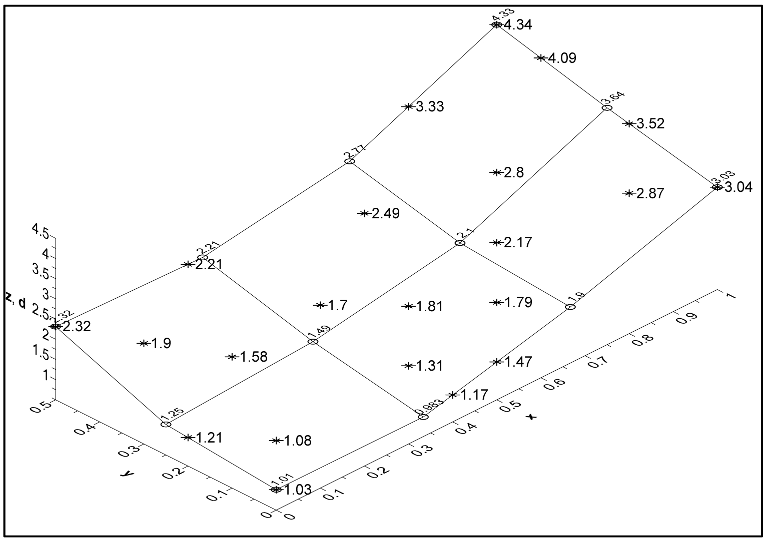
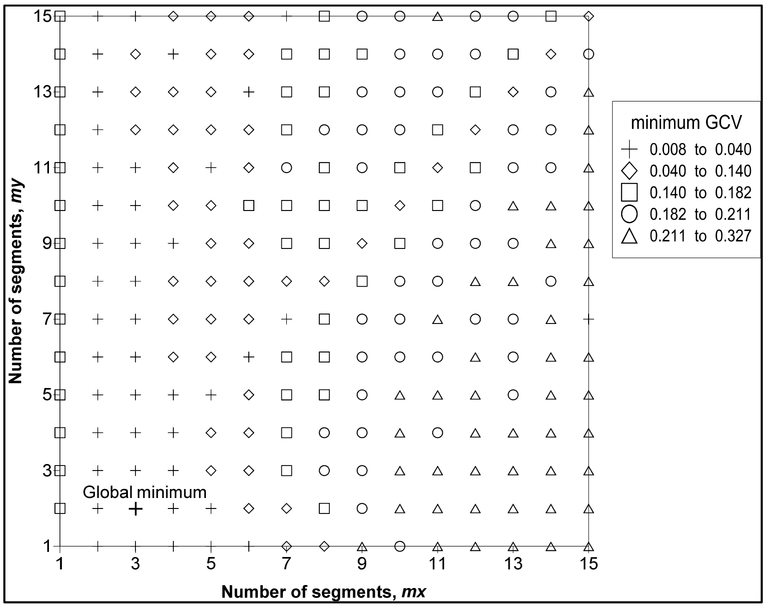


Fig. 3 Bilinear surface *d* (circles) fitted to the 22 data points (stars) derived from function *z* (minimum GCV: mx = 3, my = 2).



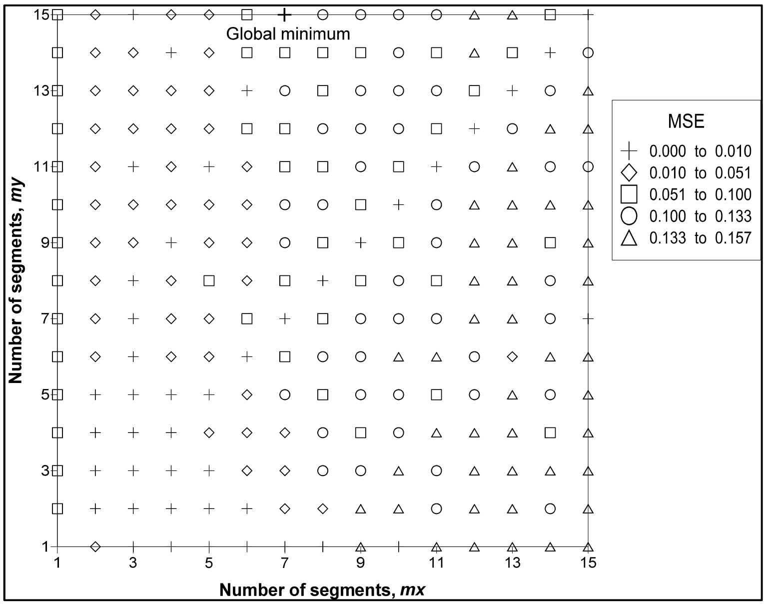


Fig. 4 Variation of the minimum GCV and of the corresponding MSE, versus the number of my, mx segments (global minimum GCV and MSE at mx = 3, my = 2).

Figure 4 depicts the variation of the minimum GCV and the corresponding MSE versus all possible combinations of segments mx, my. The location of the global minimum for GCV is placed at mx = 3, my = 2, while minimum MSE is placed at mx = 7, my = 15. Also, Fig. 4 confirms that the proposed mathematical formulation ensures the presence of a single global minimum value of GCV according to equation (10) and therefore the applicability of the objective way to assess the optimal values of adjustable parameters, as it was previously noted.

When GCV is minimized the two indices follow similar patterns, with the most characteristic one to be the variation along mx = my, where GCV's small values are encountered. Also, the similarity along the patterns of the two indices along axis *y*, for optimal number of segments on the *x* axis (mx = 3, Fig. 5) is obvious and respectively similar are the patterns of the two indices along axis x, for optimal number of segments on the Y axis (my = 2, Fig. 6).

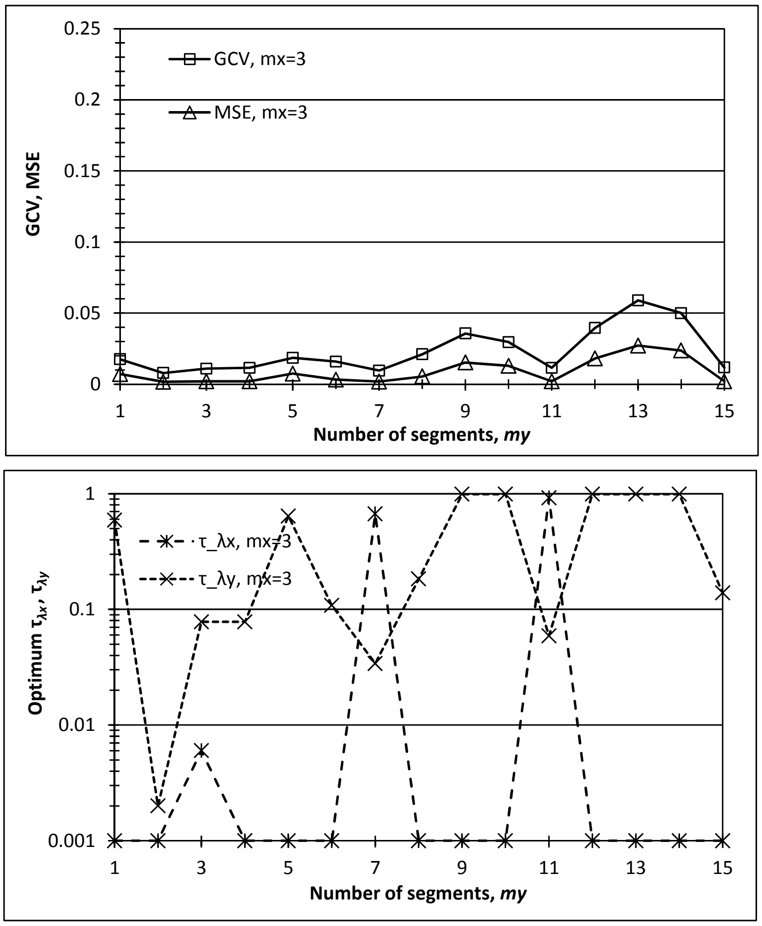


Fig. 5 Variation of the minimum GCV and corresponding MSE values along with the variation of the smoothing parameters $\tau_{\lambda x}$, $\tau_{\lambda y}$, versus the number of segments, my, for the optimal number of segments mx (global minimum GCV: mx = 3, my = 2).

Figures 5 and 6 present the variation of $\tau_{\lambda x}$ and $\tau_{\lambda y}$ optimal values along axis y, for optimal number of segments at x (mx = 3) and, likewise, the variation of $\tau_{\lambda x}$ and $\tau_{\lambda y}$ optimal values along axis x, for optimal number of segments at y (my = 2). Even though the scale is different, the pattern of $\tau_{\lambda y}$ is similar to these of the error indices in the case of retaining a constant number of segments along axis x, while the pattern of $\tau_{\lambda x}$ is similar to these of the error indices in the case of retaining a constant number of segments along axis y.

This fact constitutes a direct analogy between the proposed methodology and the one-dimensional method by Koutsoyiannis (2000) and Malamos and Koutsoyiannis (2014) since the increase, beyond a certain point, of the segments' number along one axis results in almost constant values of the smoothing parameter that refers to the opposite axis. On the other hand, the overall behaviour of the BSS method is different, due to the implementation of two-dimensional minimization procedure, as shown in Fig. 4.

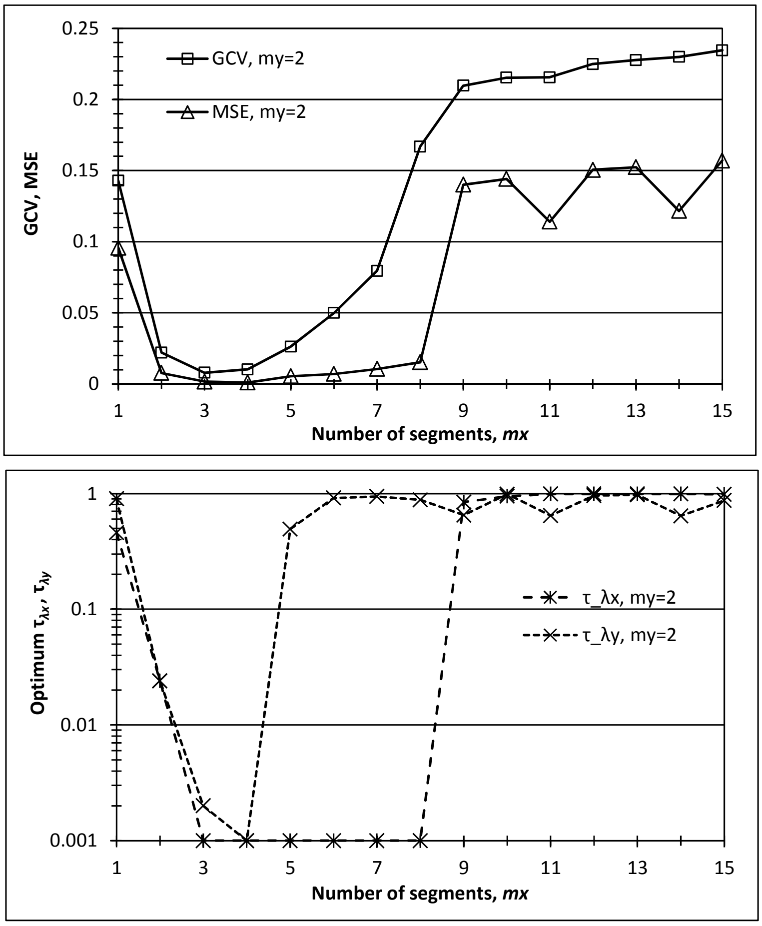


Fig. 6 Variation of the minimum GCV and corresponding MSE values along with the variation of the smoothing parameters $\tau_{\lambda x}$, $\tau_{\lambda y}$, versus the number of segments, mx, for the optimal number of segments my (global minimum GCV: mx = 3, my = 2).

Figure 7 presents the results obtained by the bilinear surface smoothing interpolation (BSS) method, using twenty-two data points (i = 22) to estimate a total of 231 points derived from the generating function described by equation (12). Also, a graphical representation of equation (12) in terms of filled contours is incorporated in Fig. 7.

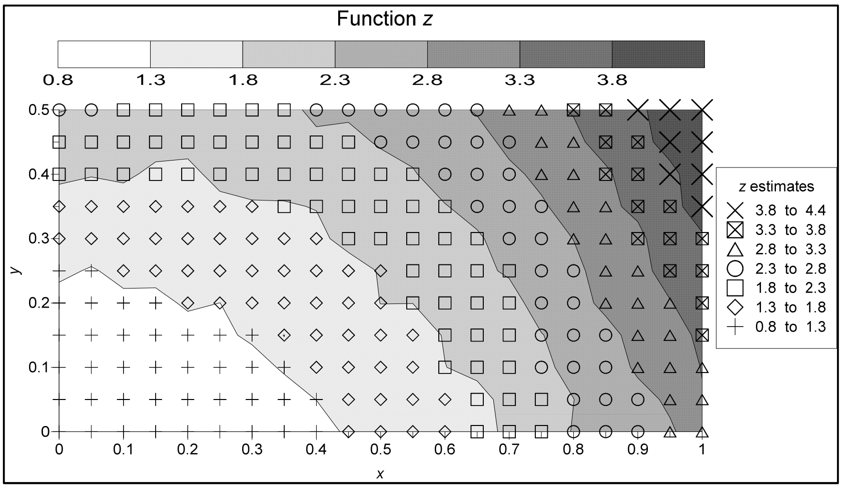


Fig. 7 Bilinear surfaces interpolation estimates of function *z* (symbols), along with the generating function.

The performance indices presented in Table 2 confirm the good performance of the BSS method. Notable is the excellent modelling efficiency, EF (see Appendix A for definition) which was obtained initially with respect to the available data points and in a second case with respect to the entire data set, derived from the generating function described by equation (12). In both cases, EF exceeded the value of 0.99, which is very close to its maximum value, that is, 1. It is apparent that the estimates are almost indistinguishable from the generating function, which suggests that the error is negligible.

Table 2 Values of performance indices used for the BSS evaluation.

Number of points and origin of the evaluation dataset	MBE	MSE	EF
22 (data points)	0	1.63×10^{-3}	0.998
231 (generating function)	1.84×10^{-2}	5.83×10^{-3}	0.991

Interpolation using Bilinear Surfaces Smoothing with the incorporation of explanatory variable (BSSE)

Since the mathematical formulation presented above allows the incorporation of an explanatory data set, $t(x_i, y_i)$, we utilized for this purpose 231 points derived from equation (13). These points formed a square grid with the same dimensions as the analysis grid. Consequently, we obtained 231 point estimates of the generating function. After the implementation of the iterative procedure for acquiring the global minimum of GCV as described previously, we obtained the optimal values of the six adjustable parameters: the number of intervals, mx, my, and the smoothing parameters λ_x, λ_y and μ_x and μ_y as presented in Table 3.

The optimal values of the $\tau_{\lambda x}$ and $\tau_{\lambda y}$ smoothing parameters concerning bilinear surface *d* are similar to those of the previous case. The optimal value of the smoothing parameter $\tau_{\mu y}$ reached the maximum allowed value used during the minimization procedure, suggesting that the optimal solution of the problem required that the difference of slopes between the consecutive segments of the bilinear surface *e* according to *y* direction should be as large as possible.

Table 3 BSSE parameters optimal values and performance indices, for the exploration application.

number of segments, mx	number of segments, my	$\tau_{\lambda x}$	$\tau_{\lambda y}$	$\tau_{\mu x}$	$\tau_{\mu y}$	MSE	Global minimum GCV
4	2	0.001	0.006	0.769	0.99	2.41×10^{-4}	3.70×10^{-3}

Figure 8 presents the bilinear surfaces *d* and *e* acquired from the solution of equation (11), along *x* and *y* axes, by using the above presented parameters. The open circles represent the values of vectors ***d*** and ***e***. The consistency to the mathematical framework is verified by the obvious difference of slopes between the consecutive segments of the bilinear surfaces according to both directions.

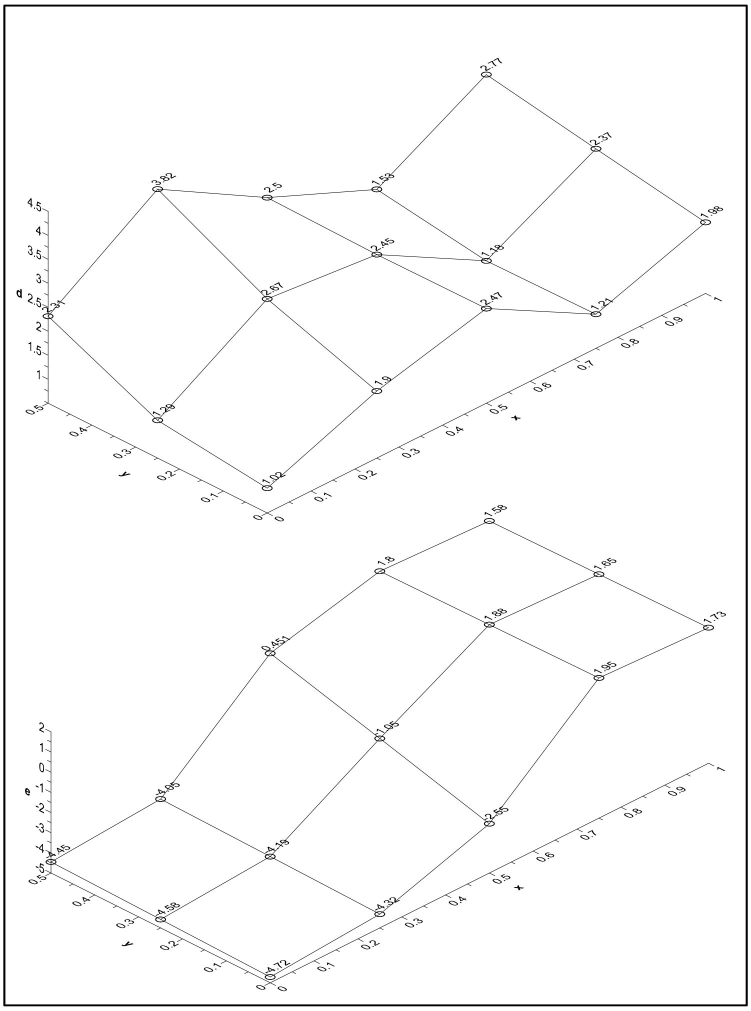


Fig. 8 Acquired bilinear surfaces *d* and *e*, so that $z = d + t\ e$ fits the 22 data points derived from the generating function (minimum GCV: mx = 4, my = 2).

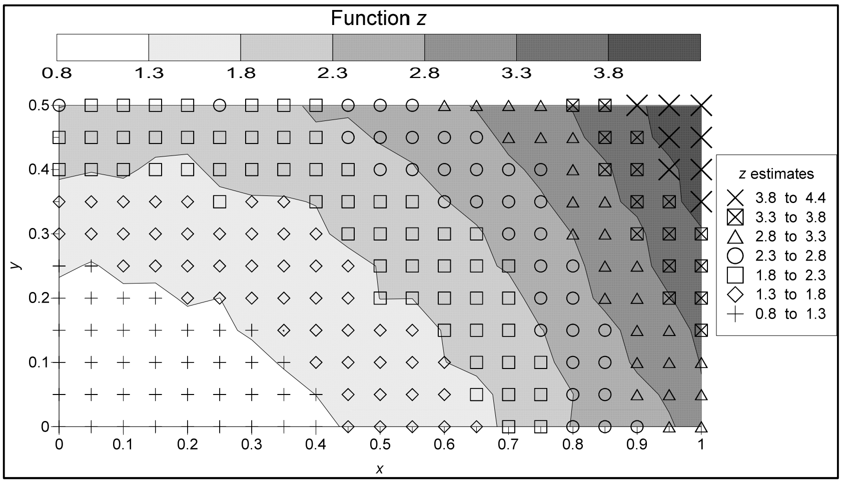


Fig. 9 Bilinear surfaces interpolation estimates of function *z* (symbols), with the incorporation of the explanatory variable *t*, along with the generating function.

Figure 9 depicts the 231 points obtained by the BSSE method, using twenty-two data points (i = 22) against the generating function described by equation (12). Also, a graphical representation of equation (12) in terms of filled contours is incorporated in Fig. 9.

The performance indices presented in Table 4 confirm the good performance of the BSSE method to incorporate the influence of the explanatory variable (Fig. 2) in the results. The negligible discrepancies between the true values and the estimates are mainly located in areas where the explanatory function $t(x, y)$ has low values. Nevertheless, the overall method performance implies its capability to perform complicated interpolation tasks. The modelling efficiency (EF) is very high, similar to the previous example, exceeding 0.98.

Table 4 Values of performance indices used for the BSSE evaluation.

Number of points and origin of the evaluation dataset	MBE	MSE	EF
22 (data points)	0	$2.41{\times}10^{-4}$	1
231 (generating function)	$2.35{\times}10^{-2}$	$8.23{\times}10^{-3}$	0.987

Real world application

For real world application we implemented both proposed versions of the methodology into spatial interpolation of over-annual rainfall. The objective of the application was: a) to verify the method’s applicability against a hydrological variable with significant correlation to an easily measurable, hence available at considerably higher resolution, explanatory variable, b) to verify the method’s versatility in terms of handling extensive datasets and c) to compare the results with commonly used methodologies like: Inverse Distance Weighted, Spline, Ordinary Kriging and Ordinary Cokriging (Goovaerts 1997, Goovaerts 2000, Burrough and McDonnell 1998, Li and Heap 2008). The above methods form a representative set for comparisons of the Bilinear Surface Smoothing methodology as they range from the simple and deterministic Inverse Distance Weighted to the more complex and stochastic Cokriging.

The study area was the region of Central Greece (Sterea Hellas; Fig. 10). The data consisted of the mean rainfall at a network of 71 meteorological stations, derived from all available measurements starting from 1992 and backwards until 1931 (Christofides and Mamassis 1995). For the majority of the stations, the available timeseries were at least 30 years long. The analysis extend (mask) boundaries were defined by the coordinates of the outermost stations according to each one of the four cardinal directions. This was mandatory in order to ensure that the rainfall estimates adjacent to the boundaries of the study area are obtained from interpolation rather than extrapolation.

Since spatial variability of precipitation, in over-annual scale, is influenced by orography (Hevesi *et al*. 1992a, 1992b, Goovaerts 2000) the topographic elevation can be used as explanatory variable for implementing the BSSE and Cokriging methodologies. The explanatory dataset was obtained from the Digital Elevation Model (DEM) SRTM Data Version 4.1 (Jarvis *et al.* 2008) and aggregated to a 2×2 km square grid (Fig. 10) for practical and computational reasons, covering approximately an area of 23 920 km^2 with 5980 points of known elevation.

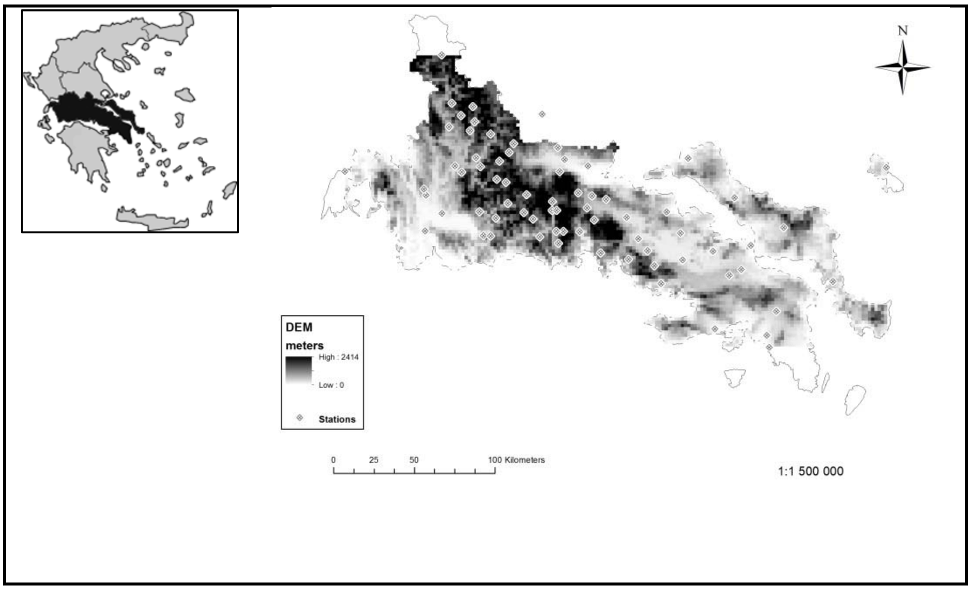


Fig. 10 Elevation map and meteorological stations of Central Greece (Sterea Hellas)

The global minimum of GCV, for both cases, was reached by implementing both proposed methodologies for different numbers of segments mx and my ($1 \leq$ mx ≤ 15 and $1 \leq$ my ≤ 15, while $m + 1 \geq 6$) and minimizing GCV for each one, by altering the adjustable parameters.

Additionally, we assessed larger values of mx and my up to 30 segments in either direction (i.e. $16 \leq$ mx ≤ 30 and $16 \leq$ my ≤ 30), by setting the smoothing parameters to their minimum value (i.e. $\tau_{\lambda x} = \tau_{\lambda y} = 0.001$ and where applicable $\tau_{\mu x} = \tau_{\mu y}$

= 0.001) in order to reduce the computational effort required to implement the GCV minimization procedure. This approach was based on the observations made by Koutsoyiannis (2000) and Malamos and Koutsoyiannis (2014), concerning the relation between large numbers of broken line segments and the minimum values of the smoothing parameters. This behaviour can be explained from the fact that increased numbers of bilinear surface segments contribute to the overall surface smoothness, thus acting as additional smoothing parameters. The results of the above procedure are presented in Table 5.

Table 5 BSS and BSSE optimal parameter values and performance indices, for the rainfall interpolation example.

Method	number of segments, mx	number of segments, my	$\tau_{\lambda x}$	$\tau_{\lambda y}$	$\tau_{\mu x}$	$\tau_{\mu y}$	Global minimum GCV
BSS	7	23	0.001	0.001	-	-	6.19×10^4
BSSE	4	8	0.965	0.04	0.946	0.606	4.96×10^4

Inverse Distance Weighted (IDW), Spline, Ordinary Kriging (OK) and Ordinary Cokriking (OCK) were performed by means of ESRI's ArcGIS environment. For the case of Spline interpolation, tension spline type (Franke 1982, Mitáš and Mitášová 1988) was implemented due to the smoothing term approach which is relevant, but not similar, to the proposed mathematical framework of bilinear surfaces interpolation. After investigation between several weight values, a weight value of 10 was utilized. According to literature, a spherical semivariogram was fitted using regression, in order to minimize the weighted sum of squares between experimental and model semivariogram values (Goovaerts 2000). The results were similar to those already presented in the study of Koutsoyiannis and Marinos (1995), since the occurring discrepancies between the cokriging implementations, were related to the use of different digital elevation models.

Figure 11 presents the rainfall surface obtained from BSS along with the corresponding results from BSSE, while Fig. 12 presents the rainfall surfaces obtained from IDW, Spline, Ordinary Kriging and Ordinary Cokriging interpolation techniques.

A clear west-east rainfall gradient is apparent in all cases, with high precipitation in the west due to the greater influence of the Ionian Sea (west of the area). The influence of the Aegean Sea (east of the area) is clear in the north-east part of the maps.

The performance of each method (Table 6) was evaluated by using statistical criteria such as: mean bias error (MBE), mean absolute error (MAE), root mean square error (RMSE), mean square error (MSE) and modelling efficiency (EF) which is calculated on the basis of the relationship between the observed and predicted mean deviations (Willmott 1982, Vicente-Serrano *et al.* 2003, Li and Heap 2008). The relationships that provide them are depicted in Appendix A (Equations A1 to A5).

Table 6 Values of the statistical criteria used to assess the performance of the spatial interpolation methods.

Interpolation method	MBE (mm)	MAE (mm)	RMSE (mm)	MSE	EF
BSS	0.0	129.3	172.9	3.0×10^{4}	0.82
BSSE	0.0	140.1	185.7	3.4×10^{4}	0.80
IDW	–0.2	5.3	9.4	8.8×10^{1}	1.00
Spline	3.5	11.8	18.4	3.4×10^{2}	1.00
Ordinary Kriging	1.5	70.3	88.7	7.9×10^{3}	0.95
Ordinary Cokriging	3.5	86.5	110.7	1.2×10^{4}	0.93

From Table 6 it is apparent that IDW, Spline and both kriging methods, according to the statistical criteria used, outperform the bilinear surface smoothing methods, apart from the MBE criterion. This is not a surprise because:

(a) Kriging, from construction, minimizes the MSE.
(b) IDW is an exact method of interpolation, so its results respect the data points exactly.
(c) Spline is forced to pass "not too far" from the data points (Burrough and McDonnell 1998).

Nevertheless, the results of both bilinear surface smoothing methods are very satisfactory.

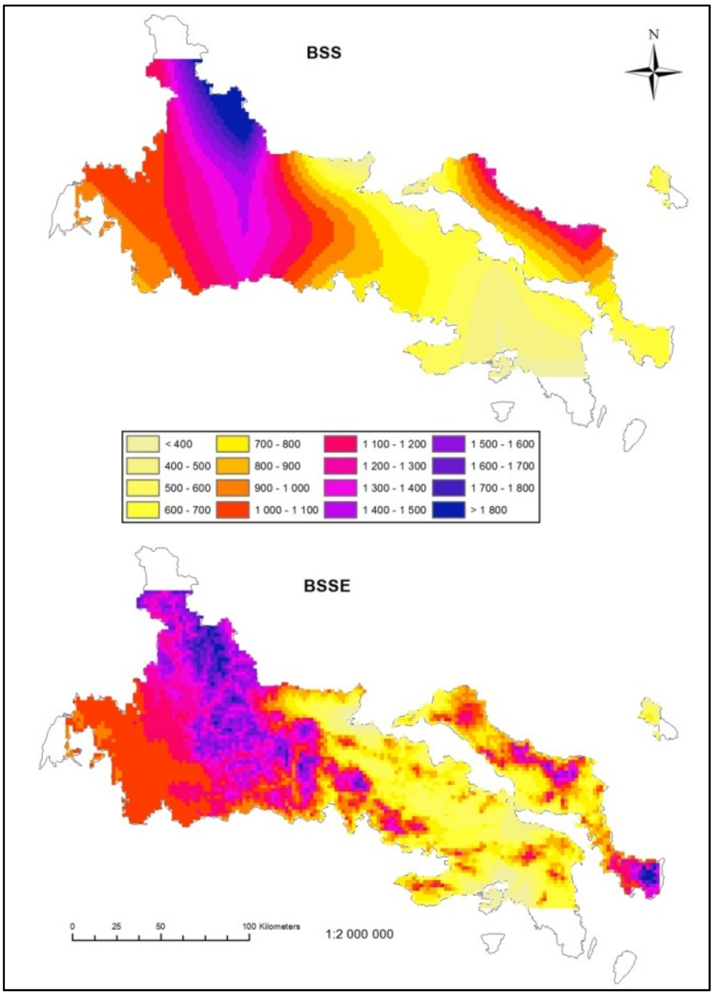


Fig. 11 Rainfall maps (mm) produced by Broken Surface Smoothing (BSS) and Broken Surface Smoothing with explanatory variable (BSSE).

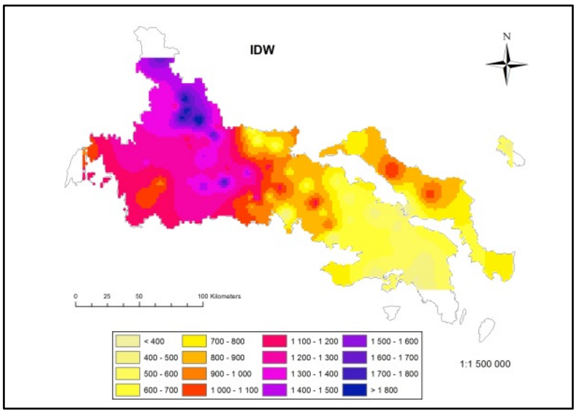
Daly *et al*. 2002, emphasized that the human factor, in terms of expert knowledge on the spatial patterns of climate in a specific region, is capable of enhance, control, and parameterize computer based interpolation techniques. Based on that principle, our interpretation of rainfall spatial patterns (Fig. 11, 12) suggests that both cases of bilinear surface smoothing respect in a more efficient way the dependence of rainfall on the west-east gradient and the elevation (with increased elevation, rainfall increases as happens in reality).

Also, the above statistical criteria performance may not be representative with respect to the validity of the interpolation results in other locations, except for those incorporated in the interpolation procedure. So, an alternative technique was implemented for the evaluation of the bilinear surface smoothing methods efficiency, in terms of performing validation between two subsets of the available data. The first acts as input to each one of the four interpolation methods while their outcome is compared against the second subset.

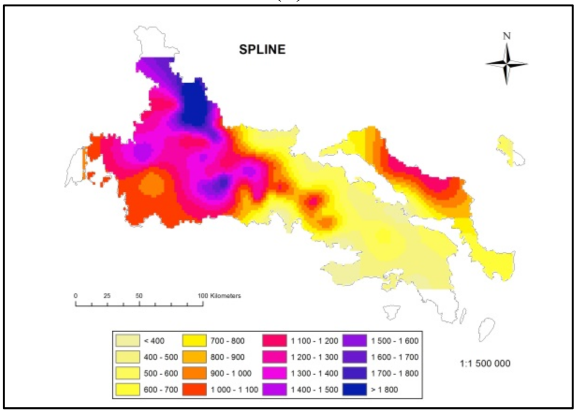
In this context, while keeping the same analysis extent and boundaries, we divided randomly, the 71 meteorological stations network of the study area, into a subset that comprised 29 meteorological stations and acted as input dataset, while the second subset contained the remaining 42 stations and acted as validation dataset. The implementation of the interpolation procedures followed the previously presented approach and the results concerning BSS and BSSE, are presented in Table :

Table 7 BSS and BSSE parameters optimal values and performance indices, for the rainfall interpolation example validation procedure.

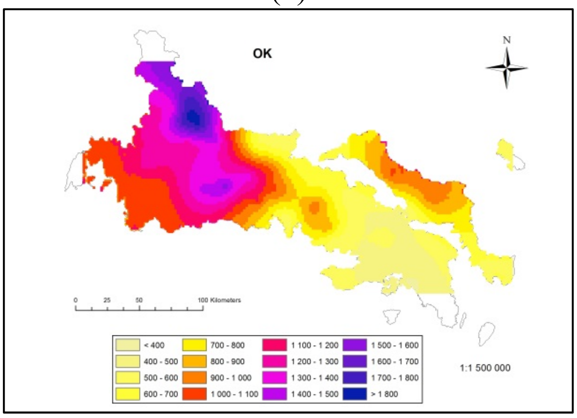
Method	number of segments, mx	number of segments, my	$\tau_{\lambda x}$	$\tau_{\lambda y}$	$\tau_{\mu x}$	$\tau_{\mu y}$	Global minimum GCV
BSS	13	14	0.01	0.01	-	-	5.68×10^{4}
BSSE	13	14	0.697	0.01	0.845	0.913	3.20×10^{4}



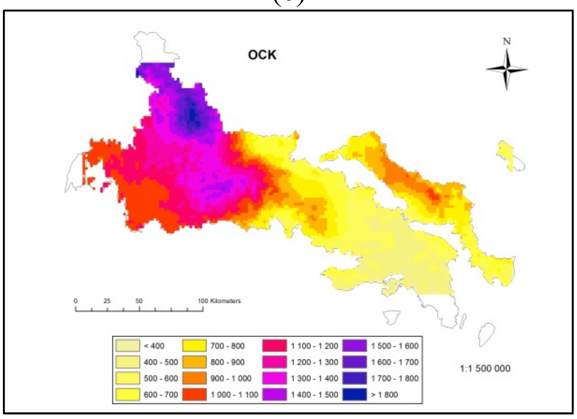
(a)



(b)



(c)



(d)

Fig. 12 Rainfall maps (mm) produced by (a) IDW, (b) Spline, (c) Ordinary Kriging (OK) and (d) Ordinary Cokriging (OCK).

Figure 13 presents the rainfall surface obtained from BSS methodology along with the corresponding results from BSSE methodology, while Fig. 14 presents the rainfall surfaces obtained from IDW, Spline, Ordinary Kriging and Ordinary Cokriging interpolation techniques.

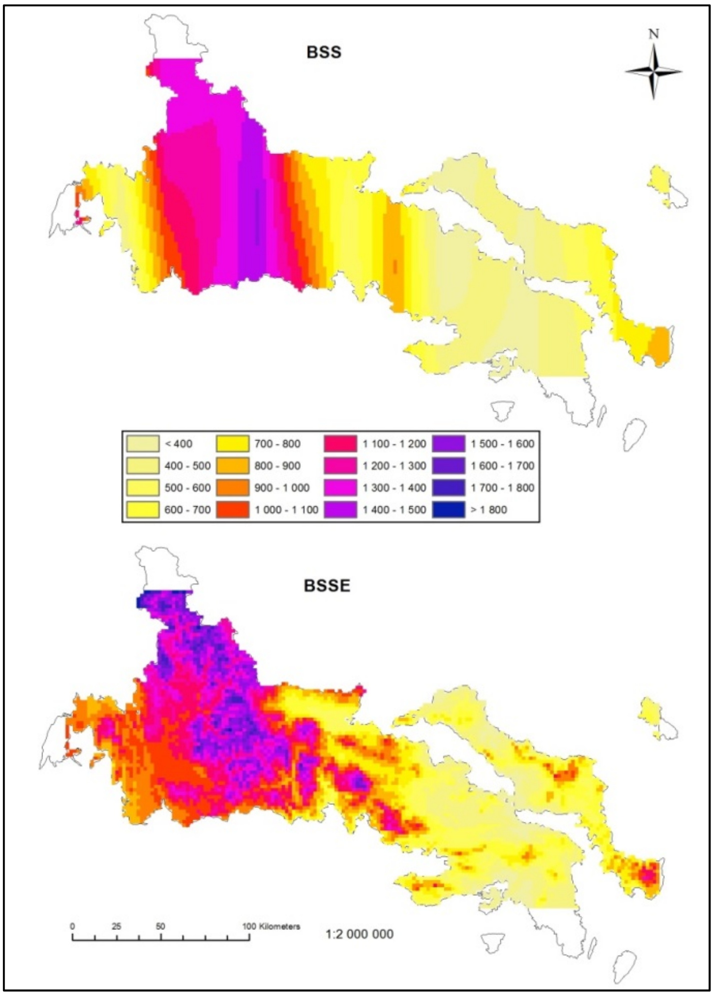
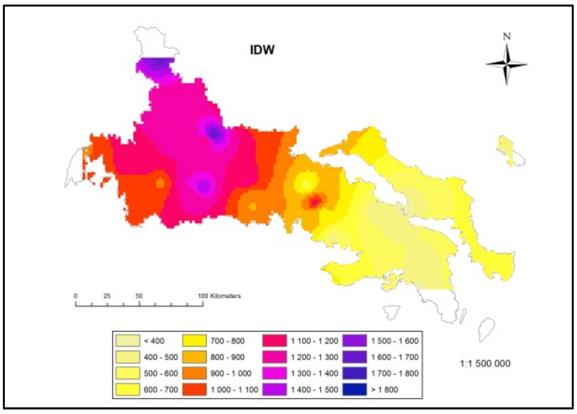
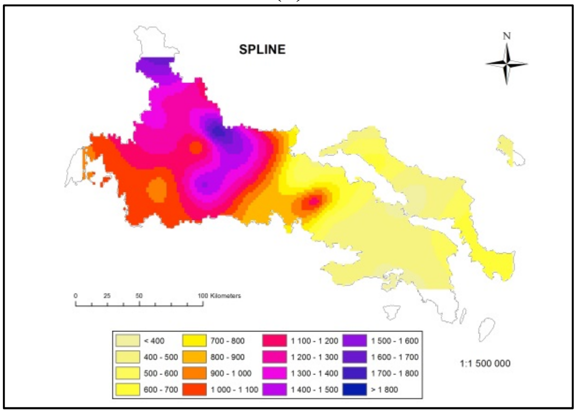


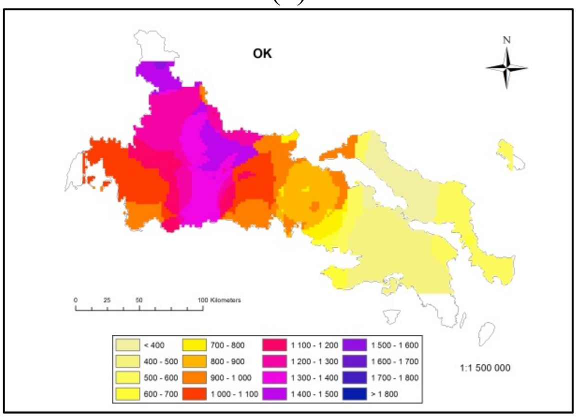
Fig. 13 Rainfall maps (mm) produced by Broken Surface Smoothing (BSS) and Broken Surface Smoothing with explanatory variable (BSSE) for the validation procedure (29 of 71 meteorological stations available).



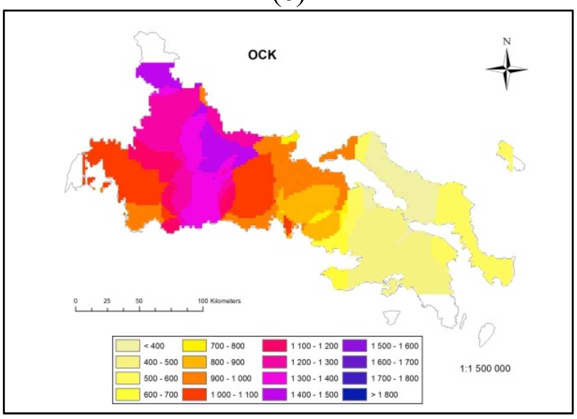
(a)



(b)



(c)



(d)

Fig. 14 Rainfall maps (mm) produced by (a) IDW, (b) Spline, (c) Ordinary Kriging (OK) and (d) Ordinary Cokriging (OCK) for the validation procedure (29 of 71 meteorological stations available).

Visual interpretation of both figures indicates that the broken surface smoothing methodology produced satisfactory results even though a limited amount of data was available. Especially the BSSE version of the methodology, as shown in Fig. 13, produced a very plausible interpolation surface that respects the variation due to orography and the west-east rainfall gradient, in contrast to IDW, Spline and both kriging methods.

In order to establish how the proposed interpolation methodology preserves the stochastic characteristics (first and second statistical moments) of the interpolated field, we present in Table 8 the mean values along with the standard deviations of the results acquired by the six methods against the stations data, for the validation case.

Table 8 Mean value and standard deviation of the results of the spatial interpolation methods against the stations data, in the validation case.

	Mean value (mm)	Standard deviation (mm)
Stations data	1077.6	429.0
BSS	907.7	395.2
BSSE	987.1	379.4
IDW	1021.1	259.9
Spline	1008.0	351.7
Ordinary Kriging	1026.5	316.1
Ordinary Cokriging	1025.6	316.5

All interpolation methods present negative bias both in mean and standard deviation, with the BSS and BSSE corresponding to the highest bias for the mean and the lowest bias for the standard deviation, thus representing the variability of the rainfall field better than the other methods.

Additionally, Table 9 summarizes the values of the statistical criteria acquired with respect to the second subset. BSSE clearly outperformed IDW, Spline and both kriging methods in estimating the mean annual rainfall measured at the 42 meteorological stations, apart from the MBE criterion. BSS outperformed Spline and both kriging methods and performed similarly to IDW, apart from the MBE criterion.

Table 9 Values of the statistical criteria used to assess the performance of the spatial interpolation methods, in the validation case.

Interpolation method	MBE (mm)	MAE (mm)	RMSE (mm)	MSE	EF	IPE3
BSS	–170.0	255.0	323.0	10.4×10^4	0.42	0.86
BSSE	–90.5	195.2	265.5	7.1×10^4	0.61	0.62
IDW	–56.5	244.3	318.1	1.2×10^5	0.44	0.74
Spline	–69.7	262.9	346.3	1.0×10^5	0.33	0.82
Ordinary Kriging	–52.0	263.4	345.4	11.9×10^4	0.34	0.82
Ordinary Cokriging	–51.2	266.5	348.5	12.1×10^4	0.32	0.83

Apart from the above presented criteria, an inter-comparison technique in terms of an Ideal Point Error (IPE) which is calculated by identifying the ideal point to a multi-dimensional space that each model should be evaluated against (Domínguez *et al*. 2011), was implemented in order to demonstrate the performance of the proposed methodology against the other four methods. The comparison was based on the use of a combined evaluation vector comprising from three traditional metrics, as shown in Appendix A (Equation A6). The acquired values of the IPE3 criterion presented in Table 9 verify that BSSE outperformed all other methods, while BSS performed similarly to them.

Based on the above discourse, it is clear that the Bilinear Surface Smoothing methodology is able to perform complex interpolation tasks even in cases of scarce data sets.

CONCLUSIONS

A non-parametric spatial interpolation methodology (BSS) which approximates a surface that may be drawn for the available data points with consecutive bilinear surfaces with known break points and adjustable weights is utilized to perform various interpolation tasks. Additionally, an alternative to the main methodology (BSSE) that incorporates, in an objective manner, an explanatory variable by combining two bilinear surfaces into the same regression model, was implemented. The mathematical framework, the computational implementation and details concerning both versions of the methodology are discussed in a companion paper (Malamos and Koutsoyiannis submitted).

Both versions were illustrated and tested against two applications, a theoretical one with synthetic data from a known generating function and a real world example: the spatial interpolation of rainfall data with or without the use of surface elevation, as explanatory variable.

The interpolations performed to the synthetic data were successful by all means, either with respect to the available data points or with respect to the entire data set, according to the performance indices used, especially for BSS. The behaviour of the proposed mathematical framework was analogous to the single dimension methods presented by the authors in previous studies. This is clearly demonstrated by the variation patterns of the minimum GCV and corresponding MSE values when plotted against the number of segments of the bilinear surface.

Also, a comparison to the results of commonly used methodologies like IDW, Spline, Ordinary Kriging and Ordinary Cokriging was conducted. Additionally, for validation purposes, the original dataset was divided into two subsets. One served as input dataset, while the second subset that contained the remaining stations was the validation dataset. In every case, the methods' efficiency to perform interpolation between data points that are interrelated in a complicated manner was confirmed.

The applicability and consistency of the mathematical framework against not only dense but also scarce data sets, is supported by the fact that the method's resolution (number of consecutive bilinear surfaces) does not necessarily has to coincide with that of the given data points, but it can be either finer or coarser, depending on the specific requirements of the problem of interest. This was verified by the validation procedure presented in the real world case study, in which BSSE gave very good results outperforming those of the other interpolation methods in many aspects.

Given the simplicity of the approach, the overall performance of the proposed mathematical framework is quite satisfactory, indicating its applicability for diverse scientific and engineering tasks related to hydrology and beyond, without the need to make arbitrary decisions on parameters. The approach seems promising in all respects but further research and applications need to be conducted to investigate the strengths and weaknesses of the method.

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APPENDIX A

STATISTICAL CRITERIA

The statistical criteria used for the evaluation of the methodologies performance are: mean bias error (MBE), mean absolute error (MAE), root mean square error (RMSE), mean square error (MSE) and modelling efficiency (EF) (Nash and Sutcliffe 1970, Willmott 1982, Loague and Green 1991). Willmott (1982) suggests that RMSE and MAE are among the "best" overall measures of model performance, as they summarize the mean difference in the units of observed and predicted values. The problem is that RMSE provides a measure of model validity that places a lot of weight on high errors, whereas MAE is less sensitive to extreme values. The relationships that provide them are:

$$\mathrm{MBE} = \frac{1}{n}\sum_{i=1}^{n}(P_i - O_i), \tag{A1}$$

$$\mathrm{MAE} = \frac{1}{n}\sum_{i=1}^{n}|P_i - O_i|, \tag{A2}$$

$$\mathrm{RMSE} = \left[\frac{1}{n}\sum_{i=1}^{n}(P_i - O_i)^2\right]^{1/2} \tag{A3}$$

$$\mathrm{MSE} = \frac{1}{n}\sum_{i=1}^{n}(P_i - O_i)^2 \tag{A4}$$

$$\mathrm{EF} = 1 - \frac{\sum_{i=1}^{n}(P_i - O_i)^2}{\sum_{i=1}^{n}(\bar{O} - O_i)^2} \tag{A5}$$

where n is the number of observations, O_i are the observed values, P_i are the predicted values, while $\bar{O}$ is the mean of the observed values. The optimum (minimum) for the MBE, MAE, RMSE, MSE statistics is 0, while the optimum (maximum) for EF is 1.

Ideal Point Error

The Ideal Point Error (IPE) (Domínguez *et al*. 2011) measurement is calculated by identifying the ideal point, up to a five-dimensional space, that each model should be evaluated against. For the purposes of the present study, the three-dimensional IPE3 is implemented by normalizing RMSE, MBE and the coefficient of determination (R^2), so the individual IPE3 for each measure ranges from 0 for the best model to 1 for the worst.

The coordinates of the ideal point are: RMSE = 0, $R^2 = 1$, MBE = 0. IPE3 measures how far a model is from this ideal point by the relationship:

$$\mathrm{IPE3} = \left[0.33\left(\left(\frac{\mathrm{RMSE}_i}{\max \mathrm{RMSE}}\right)^2 + \left(\frac{R^2_i - 1}{\min R^2 - 1}\right)^2 + \left(\frac{\mathrm{ME}_i}{\max|\mathrm{ME}|}\right)^2\right)\right]^{1/2} \tag{A6}$$

In equation A6, i represents each of the models under investigation.