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Probabilistic description of rainfall intensity at multiple time scales

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1. Abstract

The probabilistic description of the average rainfall intensity over a certain time scale in relationship with the time scale length has theoretical interest, in understanding the behaviour of the rainfall process, and practical interest in constructing relationships between intensity, time scale (sometimes called "duration") and return period (or "frequency"). To study these relationships, the principle of maximum entropy can serve as a sound theoretical background. Using a long rainfall dataset from Athens, Greece, and time scales ranging from 1 hour to 1 year, we study statistical properties such as (a) probability dry and its relationship with rainfall intensity and time scale, (b) marginal probability distribution function of rainfall intensity, with emphasis on the tails, and its variation with time scale (c) dependence structure of rainfall intensity with reference to time scale, and (d) statistical properties that are invariant or scaling with time scale. The study concludes with a discussion of the usefulness of these analyses in hydrological design.

2. The data set

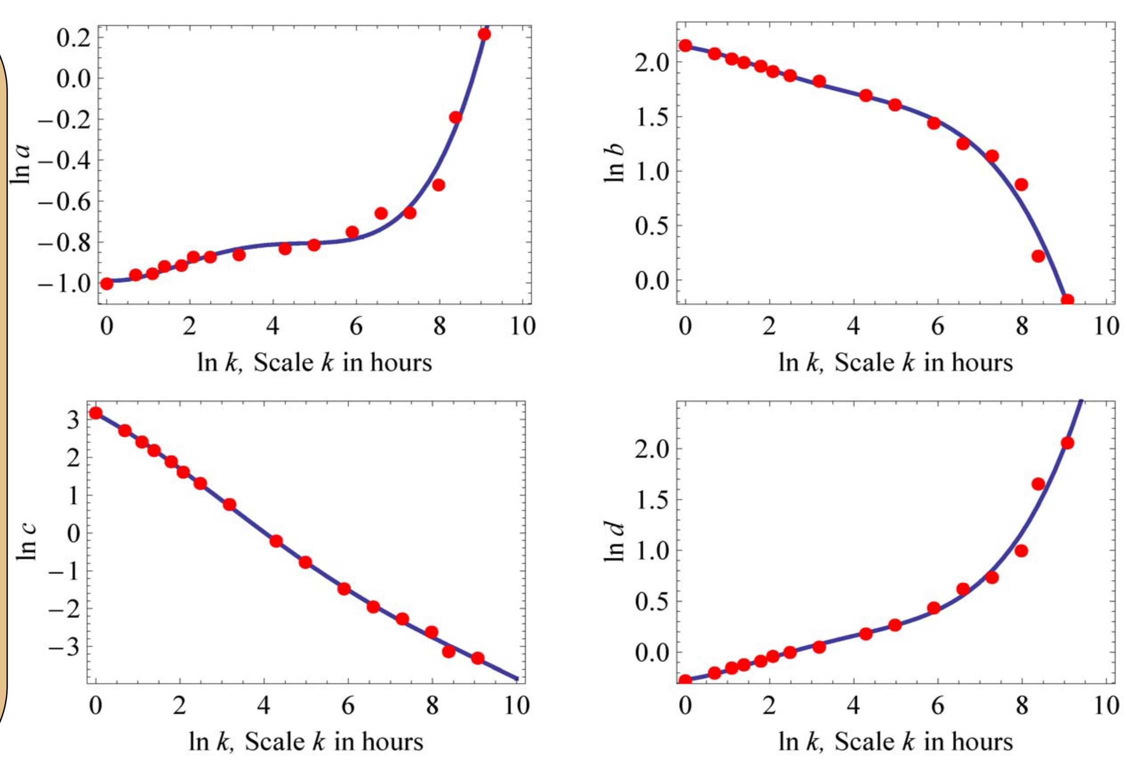
The data set is a 70-year long hourly rainfall record from the station of National Observatory of Athens, Greece. The table presents statistics of rainfall intensity averaged over several time scales.

Scale	P_{Dry}	Mean	St.Dev.	C_v	C_s	C_k	Max
1hr	0.94	0.703	1.71	2.43	7.36	106.92	58.56
2hr	0.93	0.580	1.30	2.24	6.53	87.51	38.25
3hr	0.92	0.513	1.08	2.10	6.34	96.71	33.98
4hr	0.91	0.449	0.92	2.05	5.89	82.08	26.71
6hr	0.90	0.377	0.74	1.96	5.47	69.00	19.25
8hr	0.88	0.327	0.62	1.89	4.73	47.19	13.36
12hr	0.86	0.276	0.50	1.80	5.25	67.89	12.15
24hr	0.79	0.185	0.31	1.69	4.24	40.41	6.08
≃3d	0.62	0.101	0.15	1.46	3.38	22.50	2.00
≃6d	0.47	0.072	0.09	1.32	3.28	22.93	1.20
≃15d	0.26	0.051	0.06	1.13	2.53	15.11	0.62
≃1m	0.13	0.043	0.04	0.98	1.55	6.43	0.32
≃2m	0.06	0.039	0.03	0.83	1.03	4.08	0.18
≃4m	0.00	0.038	0.03	0.71	0.57	3.03	0.13
≃6m	0.00	0.040	0.01	0.36	0.64	4.10	0.09
≃12m	0.00	0.040	0.01	0.27	0.38	3.05	0.07

7. The JH distribution's parameters versus time scale

In order to construct consistent ombrian relationships it is necessary to relate the distribution's parameter values with the scale.

The parameters in log-log plots seem to vary linearly with scale when the fitted distribution remains J shaped (small scales, $ad < 1$) while, in larger scales (k > 3 months, $ad > 1$) when the distribution becomes bell shaped they change behaviour. The relationship of parameters with scale for the entire range of scales was described by polynomials of fourth degree.



4th degree fitted polynomials

$$\ln a = 0.001\ln^4 k - 0.02\ln^3 k + 0.06\ln^2 k - 0.02\ln k - 1$$
$$\ln c = -0.0006\ln^4 k + 0.02\ln^3 k - 0.1\ln^2 k - 0.6\ln k + 3.2$$
$$\ln b = -0.001\ln^4 k + 0.01\ln^3 k - 0.05\ln^2 k - 0.06\ln k + 2.1$$
$$\ln d = 0.001\ln^4 k - 0.01\ln^3 k + 0.05\ln^2 k + 0.05\ln k - 0.3$$

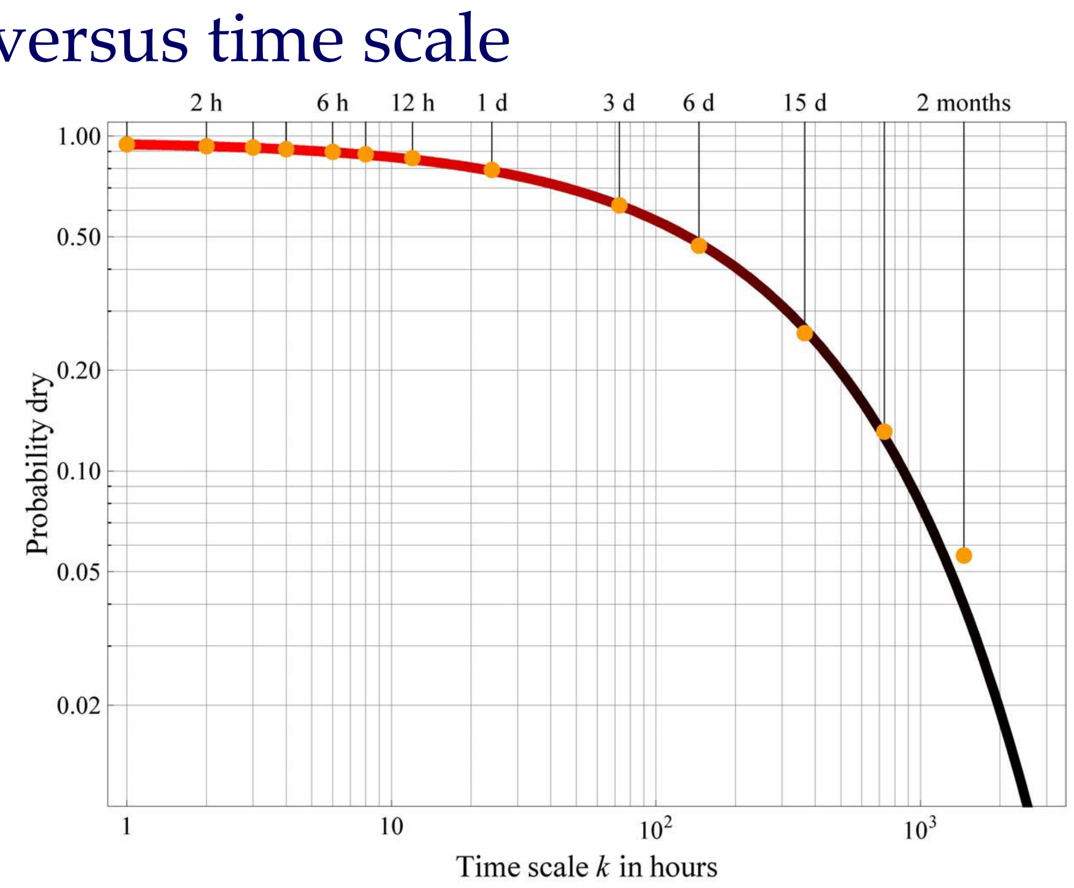
3. Probability dry versus time scale

The average rainfall intensity in a particular time scale is strongly related to the probability dry of that scale. So, it is of importance to identify any theoretical relationships relating probability dry and time scale. An investigation based on maximum entropy theoretical considerations can be found in Koutsoyiannis, 2006.

The theoretical relationship that describes the probability dry in any time scale k is given by

$$p^{(k)} = p^{\left[1+\left(\xi^{-1/\eta}-1\right)(k-1)\right]^{\eta}}$$

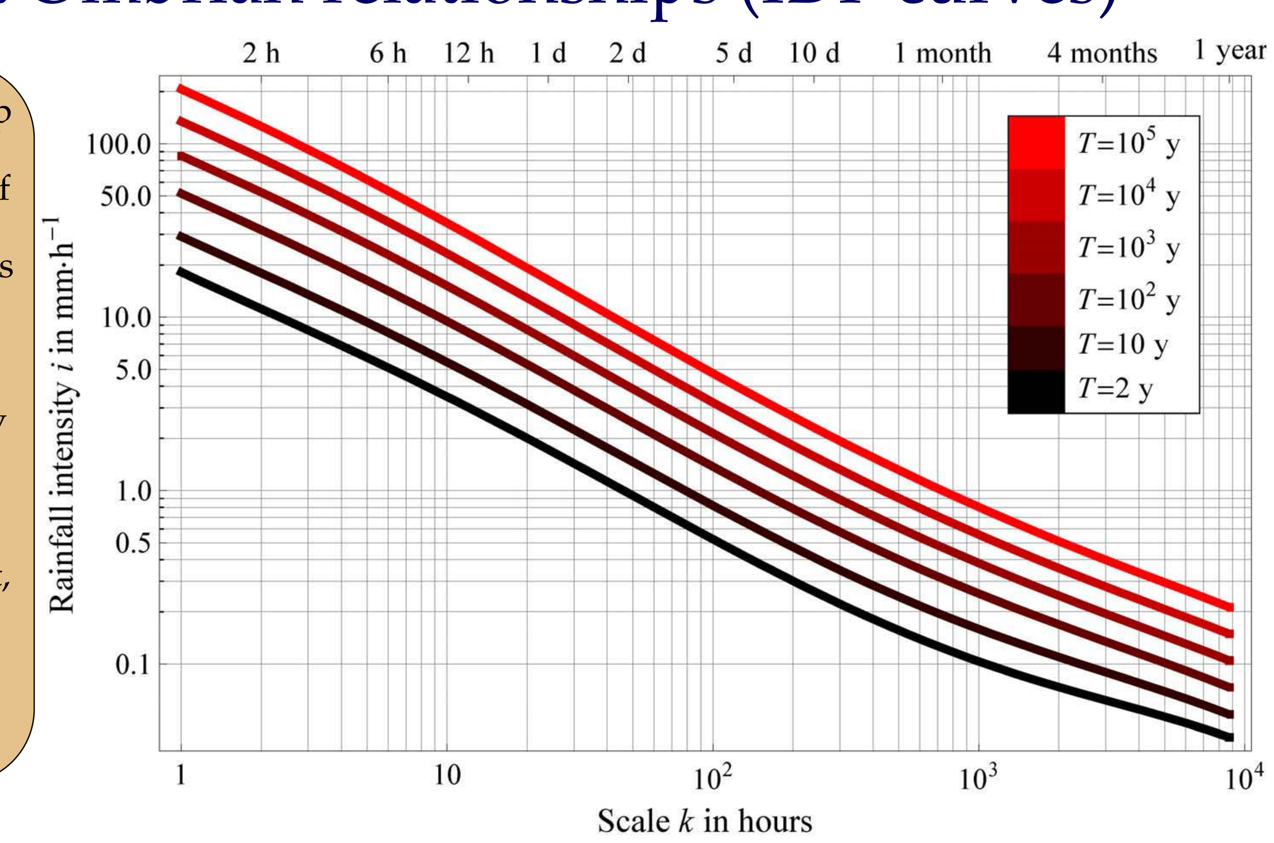
where $p = p(1)$ and ξ and η are parameters.



The figure reveals a perfect match of the fitted theoretical relationship and the empirical points.

8. Consistent Ombrian relationships (IDF curves)

An ombrian relationship expresses the rainfall intensity as a function of the return period T and the time scale k. Thus, its consistent construction presupposes to express mathematically in all scales (a) the probability distribution of rainfall intensity conditional on being wet and (b) the probability of being wet, P_{Wet}. The synthesis of these two, results in the following general expression.



$$i(k,T) = Q\left(1-\frac{k}{H\cdot P_{Wet}^{(k)}\cdot T}, a(k), b(k), c(k), d(k)\right),$$ where i in $\text{mm}\cdot\text{h}^{-1}$, k in h, T in years, Q the JH quantile, H=8766 $\text{h}\cdot\text{year}^{-1}$

4. Entropy maximization

It is well known that maximization of the standard Boltzmann-Gibbs-Shannon (BGS) entropy with simple constraints like known mean μ and variance σ^2 and the non-negativity constraint will result in an exponential type distribution which is none other than the truncated Normal distribution (see e.g. Papoulis, 1991).

In addition, maximization of the Tsallis entropy (Tsallis, 1988; 2004), a generalization of the BGS entropy, with the same simple constrains will result in a power type distribution known as the Tsallis distribution.

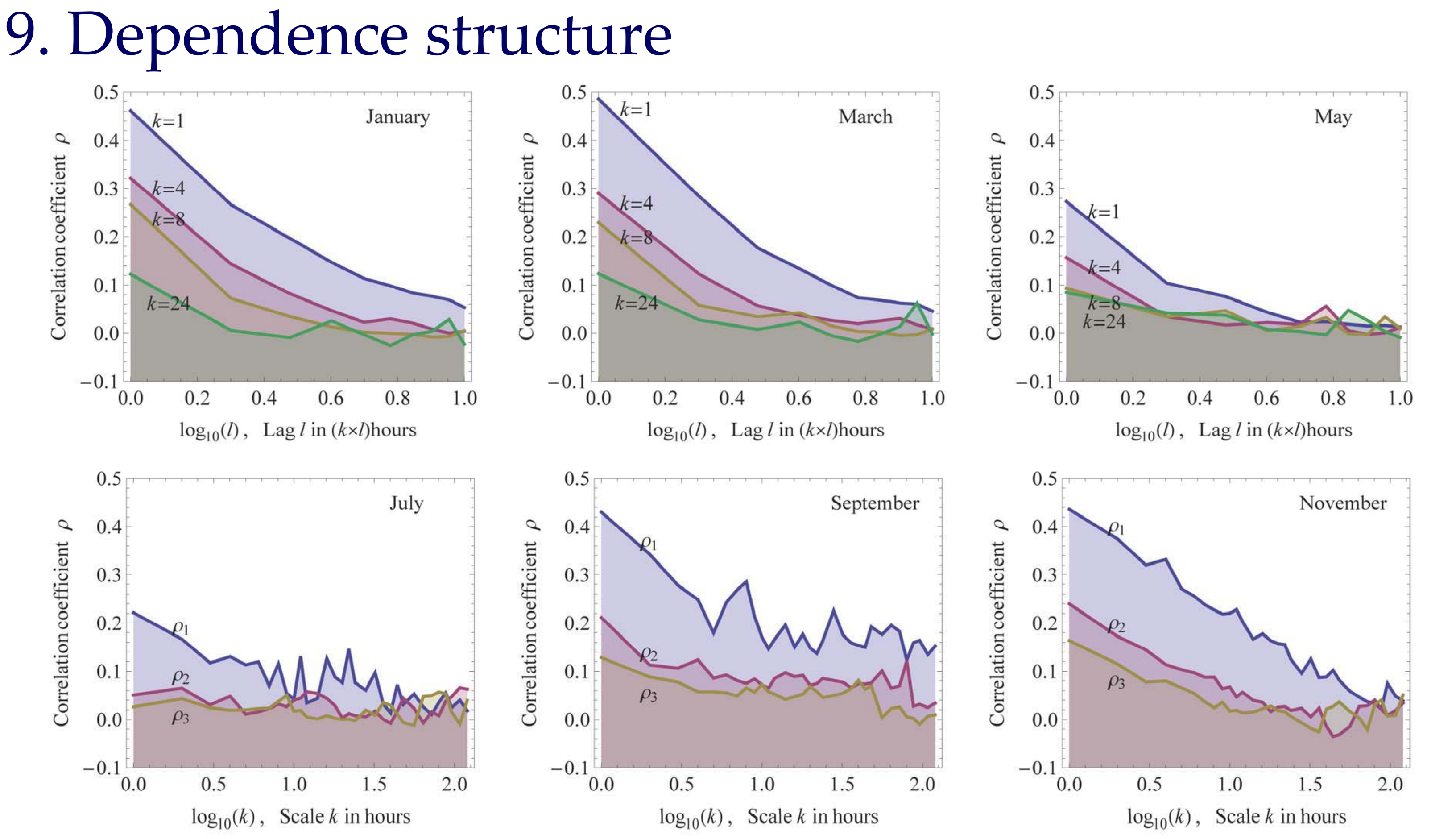
More recently, Koutsoyiannis (2008) proposed a stepwise entropy maximization approach, which also results in power type distributions. In contrast with the Tsallis distribution, that its density is bounded from above in 0, that is $f_X(0) < \infty$, the distributions produced by the stepwise maximization procedure, have the important property $f_X(0) = \infty$.

5. The JH distribution

Application of the Tsallis distribution (Papalexiou and Koutsoyiannis, 2008) showed that it fails to describe both tails of the empirical probability distribution. Thus, we propose as a rainfall distribution for all time scales, a generalization of the beta prime distribution, which we call JH distribution. The JH distribution is based on the characteristics of the distribution derived by the stepwise entropy maximization approach and in comparison is more easy to handle.

Basic characteristics of the JH distribution	
Variable range	$\mathbb{R}^+$
Constrains	$a,b,c,d \in \mathbb{R}^+$
Characteristics	$a\cdot d < 1 \Rightarrow f_X(0) = \infty$, J shaped density $a\cdot d > 1 \Rightarrow f_X(0) = 0$, Bell shaped density
Density function	$f_X(x) = \frac{d}{c\,B(a,b)}\left(\frac{x}{c}\right)^{ad-1}\left[\left(\frac{x}{c}\right)^{d}+1\right]^{-(a+b)}$
Distribution function	$F_X(x) = I_{\frac{1}{1+\left(\frac{x}{c}\right)^{-d}}}(a,b)$
q^{th} raw moment	$m_q = \frac{c^q}{B(a,b)} B\left(a+\frac{q}{d}, b-\frac{q}{d}\right)$

9. Dependence structure

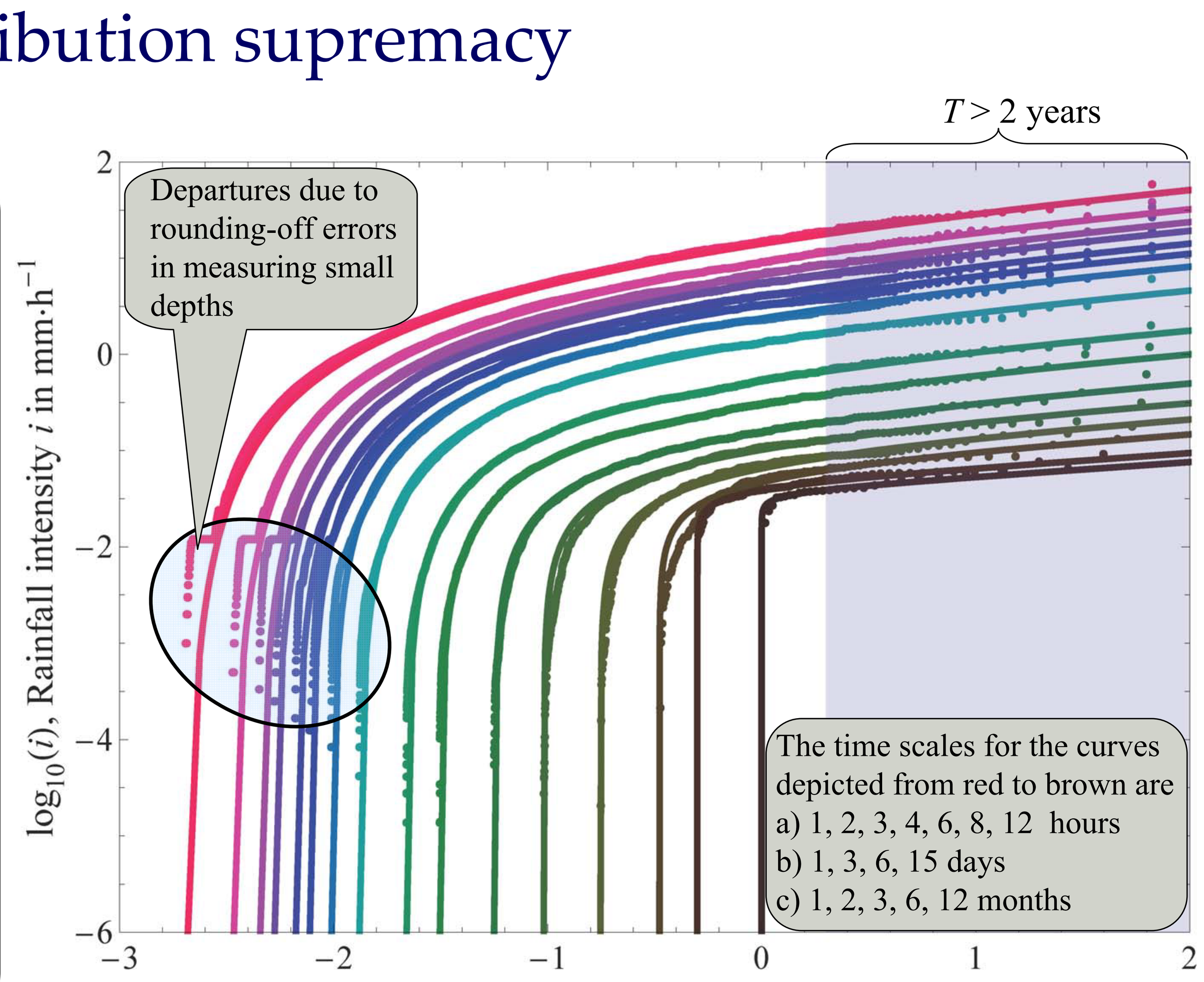


The figure does not reveal any significant long-term persistence.

6. The JH distribution supremacy

As the figure on the right attests, the JH distribution performed exceptionally well for time scales ranging from 1 h to 1 year (~4 orders of magnitude). The distribution was fitted using the method of moments. Particularly, we have used raw moments of order 0.5, 1, 1.5 and 2 to estimate the four parameters.

The examination of the region where $T > 2$ years, shows that the empirical distribution tail of each scale, is described by simple power laws with same exponent. Asymptotically, the distribution's tail behaves as x^{-bd} and in this particular data set $bd \approx 6.5$.



10. Conclusions

- We propose a new distribution for rainfall intensity based on the principle of maximum entropy that we call the JH distribution. This flexible power type distribution can be either J shaped or bell shaped and thus can be suitable for rainfall intensity both in small and large time scales. The practice revealed that the distribution performs extremely well from scales ranging from 1 hour to 1 year.
- The parameters of the JH distribution can be related to time scale by simple functions. This makes the distribution suitable for the construction of consistent ombrian relationships (also known as IDF curves).
- Using the JH distribution we derived an analytical expression relating the rainfall intensity i with the time scale k and with the return period T (ombrian relationship) for time scales ranging from 1 hour to 1 year.
- Obviously, the same distribution can yield simpler relationships if the range of the time scale is narrower.

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