Intensity-Duration-Frequency Curves from Scaling Representations of Rainfall

By

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Abstract

We develop methods to estimate the intensity-duration-frequency (IDF) curves for three rainfall models with local multifractal behavior and varying complexity. The models use the classical notion of exterior and interior process, respectively for the variation of rainfall intensity at (approximately) storm and sub-storm scales. The exterior process is non-scaling and differs in the three models, whereas the interior process is stationary multifractal in all cases. The model-based IDF curves are robust, against outliers and can be obtained from only very few years of rainfall data. In an application to a 24-year rainfall record from Florence, Italy, the models closely reproduce the empirical IDF curves and make similar extrapolations for return periods longer than the historical record.

Keywords: rainfall extremes, multifractal processes, scale invariance, IDF curves

1. Introduction

The assessment of extreme precipitation is an important problem in hydrologic risk analysis and design. This is why the evaluation of rainfall extremes, as embodied in the intensity-duration-frequency (IDF) curves, has been a major focus of theoretical and applied hydrology since the early work of Sherman (1931) and Bernard (1932); see for example the textbooks of Eagleson (1970), Chow *et al.* (1988) and Singh (1992). The IDF curves are defined as follows. Let I_d be the average rainfall intensity in a generic interval of duration d and denote by $i_{d,T}$ the value of I_d with return period T. The IDF curves are plots of $i_{d,T}$ against d for different T.

There are three basically distinct approaches to the construction of IDF curves. For T below the length of the available record, the IDF curves can be estimated directly from the yearly maximum rainfalls, using a plotting-position formula; see for example Singh (1992). This approach produces non-smooth curves, but in the few cases when a long continuous record is available (as for the case of Uccle, Belgium; see Willems, 2000) this is a viable alternative. More often, long records are available only for daily rainfall. Then the empirical IDF values for $d = 1$ day may be used to calibrate the IDF curves generated by alternative procedures or to constrain the dependence of $i_{d,T}$ on T (Koutsoyiannis, 2004a, 2004b).

A second approach, which is widely followed in practice, is to use a parametric model for $i_{d,T}$. Dependence on d is based on the typical shape of empirical IDF curves and dependence on T generally relies on the fact that rainfall maxima are attracted to extreme-type distributions. The parameters of the model are estimated from the observed annual extremes using various criteria (moment matching, maximum likelihood, least squares). For a general review and further details, see Koutsoyiannis *et al*. (1998).

A weakness of the parametric approach is that the dependence of $i_{d,T}$ on d is not based on theory. To some extent, also the dependence on T needs empirical validation, since the distribution of the yearly maxima may not conform to an extreme-value distribution and, if it does, theory alone cannot determine the appropriate asymptotic type. Perhaps a more important limitation of the parametric method is that, like the empirical approach, it uses only the historical annual maxima and neglects other information in the rainfall record. Notice in this regard that the IDF values are closely linked to the marginal distribution of the intensities I_d and significant information on this distribution is contained in the portion of the record that is being neglected.

To remedy these shortcomings, one may fit a complete model of temporal rainfall to continuous rainfall records and then use the model to generate rainfall time-series through Monte Carlo simulation; see for example Chow *et al.* (1988) and Singh (1992). Model-based IDF curves are smoother than the empirical ones and have approximate validity also beyond the range of the historical record. In addition, all the available data are used and no a-priori assumption has to be made on the shape of the IDF curves. This conceptually more satisfactory approach is rarely followed in practice because of the complexities of formulating rainfall models, estimating their parameters, and generating Monte Carlo samples.

The practical limitations mentioned above are largely due to the complex structure and extensive parameterization of rainfall models, such as those based on point processes; see for example Waymire and Gupta (1981) and Cowpertwait (1995, 1998). Here we pursue a variant of the rainfall-modeling approach in which one uses scaling (specifically, multifractal) rainfall models. This variant is capable of reproducing the intermittent character of rainfall found in many studies (Schertzer and Lovejoy, 1987, 1989; Olsson, 1995; Veneziano and Iacobellis, 2002; Venugopal *et al.,* 2006; Koutsoyiannis, 2006; among others), and as we shall show, it combines operational simplicity with accuracy and robustness due to its full use of the rainfall data.

Multifractal rainfall models (Gupta and Waymire, 1993; Lovejoy and Schertzer, 1995; Harris *et al*., 1998; Deidda, 2000; Menabde and Sivapalan, 2000; Veneziano and Langousis, 2005a, among others) are attractive for IDF analysis because they directly characterize how the distribution of I_d varies with the averaging duration d. In addition, these models are consistent with the (approximate) power-law behavior of empirical IDF curves with d and T (see for example Burlando and Rosso, 1996, and Willems, 2000). In fact, Hubert *et al*. (1998) and Veneziano and Furcolo (2002) have shown that under multifractality $i_{d,T}$ is asymptotically a power function of d and T, as $d \to 0$ for any finite range of T and as $T \to \infty$ for any finite range of d.

These scaling results are of limited practical use because they give $i_{d,T}$ up to an undetermined factor and characterize the shape of the IDF curves under asymptotic d and T conditions. In order to develop practical IDF estimation procedures, one needs to 1) formulate simple scaling models of rainfall and develop procedures to calculate the resulting IDF curves for d and T in the range of hydrologic interest, 2) if needed, devise approximations for routine application, and 3) compare the multifractal approach to current parametric methods for practicality, accuracy, robustness, data needs etc. Issue 1 is the main focus of the present paper. Issues 2 and 3 are dealt with in two fore-coming papers. The resulting procedure is simple and more accurate than the now popular parametric method. In addition, the scaling analysis sheds light on the shape of the IDF curves, in particular on the separability of the effects of d and T and the dependence of the IDF values on the return period T. These issues, which are still unsettled in the parametric approach (e.g. Koutsoyannis, 2004b) will be discussed at length in the second paper of this series. Here we anticipate that, under multifractality, separability does not strictly apply and the dependence on T, while of the power-law type for very long return periods, does not conform to an extreme distribution of Type 2.

Of course, the methods developed here apply within the scaling range of rainfall, which for most moderate climates extends from about 20-60 minutes to a few days. Departure from scaling outside this range is for example evident from the spectral analysis of rainfall records (Fraedrich and Larnder, 1993; Olsson *et al*., 1993; Olsson, 1995; Menabde *et al*., 1997; Veneziano and Iacobellis, 2002), Here we consider rainfall models that are multifractal for $d \leq D$, where D is some upper scaling limit, but impose no lower limit to scaling. It is understood that when a lower limit exists, the results should not be used beyond that limit.

The IDF curves depend somewhat on the definition of the return period T. Different definitions are possible; see for example Hubert *et al.* (1998), Willems (2000), Veneziano and Furcolo (2002) and Veneziano and Langousis (2005b). When one uses scaling rainfall models, it is mathematically convenient to take T as the reciprocal of the exceedance rate of $M_{d|D}$, the maximum of I_d in D = upper limit of multifractal behavior. Hence, if the D-periods occur with rate λ, the return period is $T(i|d) = 1/\lambda P[M_{d|D} > i]$. In calculating the distribution of $M_{d|D}$ one should consider the effects of temporal dependence of I_d inside D. However, for values of T and d of practical interest, the effect of this dependence is modest (Veneziano and Langousis, 2005b) and in what follows this effect is neglected.

Section 2 describes three stochastic rainfall models of varying complexity. Each of the subsequent three sections shows, for one model, how the model parameters are estimated and the IDF curves are calculated. To exemplify, the models are fitted to 23 years of continuous rainfall data from the Osservatorio Ximeniano in Florence, Italy (Becchi and Castelli, 1989) and the model results are compared to the empirical IDF curves for that station. Conclusions are given in Section 6.

2. Locally-Multifractal Models of Rainfall for IDF Analysis

This section describes three models of temporal rainfall. The models are tailored for IDF analysis and include simplifications that may be inappropriate for other uses. Each model is composed of an "exterior process", which characterizes the variability of rainfall at large scales, and an "interior process" for shorter-term fluctuations.

2.1 Exterior Process

The exterior processes of the three models differ as shown schematically in Figure 1. In Model 1, rainfall is produced by independent and identically distributed (iid) storms with random duration D and random average intensity I_D (Figure 1a). Storms occur at an average rate λ and are assumed to be well separated, so that at most one storm contributes to the intensity I_d. Due to our definition of the return period T, the value of λ and the joint distribution of (I_D, D) are the only characteristics of the exterior process that enter the IDF analysis. Other features of the storm arrival process are immaterial and are left unspecified. For example, storm arrival times need not be Poisson or stationary during the year.

Model 2 is simpler in that it partitions the time axis into intervals of constant duration D, where D corresponds to the upper limit of rainfall multifractality. Hence the exterior process of Model 2 is simply a sequence of consecutive rectangular pulses of constant duration and random iid intensities; see Figure 1b. Like in Model 1, not more than one D interval is assumed to contribute to the rainfall intensity I_d.

Model 3 is structurally similar to Model 2, but the mean intensity I_D in different intervals is a deterministic constant. In this case the exterior process reduces to a constant intensity I_D and a temporal scale D; see Figure 1c. Contrary to Model 2, D in Model 3 is both the outer scale of multifractal behavior and the scale at which the average rainfall intensity may be considered constant. Due to the latter condition, D in Model 3 is longer than in Model 2 and close to the interarrival time of synoptic events. In summary, in Model 1, D is both the storm duration and the upper limit of scaling of the rainfall intensity for each storm. In Models 2 and 3 there are no storms but D retains the meaning of the upper limit of the scaling range.

We emphasize that Figure 1 illustrates only the exterior component of each model. The complete model includes also the fluctuations of rainfall intensity inside the D periods, as explained next.

2.2 Interior Process

In all three models, rainfall in D (inside each storm for Model 1 and inside each D interval for Models 2 and 3) is modeled as a stationary multifractal process of the beta-lognormal type. Processes of this type have been used in the past to represent rainfall intensity; see for example Schertzer and Lovejoy (1987), Over and Gupta (1996) and Schmitt *et al*. (1998). According to this model, the average rainfall intensity I_d and its moments scale with d as

$$\begin{aligned} & I_d \overset{d}{=} A_r\, I_{rd}\,, \qquad r = \frac{D}{d} \geq 1 \\ & E[(I_d)^q] \propto d^{-K(q)}\,, \qquad q < q^* \end{aligned} \tag{1}$$

where $\overset{d}{=}$ indicates equality in distribution, A_r is the random amplitude-scaling factor of rainfall intensity for a scale-contraction factor $r > 1$ (see below), $K(q) = \log_r E[A_r^q] = C_\beta\,(q\text{-}1) + C_{ln}\,(q^2\text{-}q)$ is the associated moment-scaling function (e.g. Schertzer and Lovejoy, 1987), $q^* = \frac{1\text{-}C_\beta}{C_{ln}} > 1$ is the lowest moment order greater than 1 such that $E[(I_d)^q]$ diverges (Kahane and Peyriere, 1976), and C_β and C_{ln} are non-negative parameters that satisfy $C_\beta + C_{ln} < 1$. The parameter C_β controls the alteration of wet and dry periods within D-periods, whereas C_{ln} is responsible for the intensity fluctuations when it rains.

The random variable A_r in equation (1) has unit mean and is the product of two independent random factors: a factor $A_{ln,r}$ with lognormal distribution [specifically, ln $A_{ln.r} \sim N(\mu,\sigma^2)$, where $\mu = -C_{ln} \ln r$ and $\sigma^2 = 2C_{ln} \ln r$], and a discrete factor $A_{\beta,r}$ such that $P[A_{\beta,r} = 0] = 1 - r^{-C_\beta}$ and $P[A_{\beta,r} = r^{C_\beta}] = r^{-C_\beta}$. Hence,

$$A_r = \begin{cases} 0\,, & \text{with probability } 1\text{-}r^{-C_\beta} \\ r^{C_\beta} \exp\left(-C_{ln} \ln r + Q\sqrt{2C_{ln} \ln r}\right), & \text{with probability } r^{-C_\beta} \end{cases} \tag{2}$$

where Q is a standard normal variable. In what follows, we refer to a distribution of the type in equation (2) as a beta-lognormal distribution. The parameters C_β and C_{ln} may depend on D and I_D, but an analysis of rainfall data shows that such dependence is negligible; see Section 3.2c.

A technical point of some importance is the distinction between "bare" and "dressed" rainfall densities (Schertzer and Lovejoy, 1987). The bare density I_d' in an interval of duration $d \leq D$ is the density in d when construction of the multifractal measure through the product of fluctuations at decreasing scales is terminated at scale d, whereas the dressed measure density I_d is the average density inside d when all rainfall fluctuations, down to infinitesimal scales, are accounted for. The distributions of I_d and I_d' are related as

$$I_d \overset{d}{=} Z\, I_d' \tag{3}$$

where Z is a unit mean random variable called the dressing factor.

For binary multiplicative cascades where $r = 2^n$, Veneziano and Furcolo (2003) developed a numerical method to calculate the distribution of Z. That method requires significant numerical effort, but accurate results on multifractal extremes follow from using a simple approximation to

Z (Veneziano and Langousis, 2005b). The approximation has the form $Z \sim A_{r_Z}$ where r_Z is such that A_{r_Z} matches some moment of Z. For example, matching the moment of order q = 2 or q = 3 gives

$$
\begin{aligned}
r_Z &= \left(\frac{1}{2 - 2^{K(2)}}\right)^{1/K(2)}, \qquad && \text{for } q{=}2 \quad \text{(a)} \\
r_Z &= \left(\frac{3}{2^2 - 2^{K(3)}}\,\frac{2^{K(2)}}{2 - 2^{K(2)}}\right)^{1/K(3)}, && \text{for } q{=}3 \quad \text{(b)}
\end{aligned}
\tag{4}
$$

where $K(q)$ is the function in equation (1).With this approximation, I_d has the distribution of $I_D'A_{rr_Z}$, and estimation of the IDF curves for all three models is greatly simplified. In particular, for $d = D$, the dressed density I_D has the distribution of $I_D'A_{r_Z}$.

2.3 Objectives of Model Fitting and General Parameter Estimation Strategy

For IDF analysis, the main objective of model fitting is to accurately reproduce the distribution of the dressed densities I_d over a wide range of durations d. Special attention is given to the upper tail or higher moments of the distribution, which control the IDF curves. For all models, the parameters of the interior process are C_β and C_{ln}. The parameters of the exterior process are the joint distribution $F_{D,I_D'}$ for Model 1, the duration D and distribution $F_{I_D'}$ for Model 2, and the values of D and I_D' for Model 3.

When matching the empirical moments, we tend to use moments of order not higher than 3.5 or 4. This follows from two considerations. One is that the upper tail region of I_d that determines the IDF values for durations d and return periods T of practical interest is controlled by moments of order between about 2 and 5 (Veneziano *et al.*, 2006). The other is that, especially for d small, the empirical moments of order greater than 3 or 3.5 tend to underestimate the true moments due to sample limitations (Ossiander and Waymire, 2000, 2002). Details on the model-fitting procedure and IDF results are given in the next three sections.

3. Parameter Estimation and IDF Analysis for Model 1

This section analyzes rainfall extremes using Model 1. First we describe the method to calculate the IDF curves. Then we apply the procedure to the Florence data and in that context discuss the issues of storm extraction and parameter estimation.

3.1 IDF Analysis

Suppose for the moment that all storms have the same duration D and the same bare rainfall intensity I_D', while exhibiting internal multifractal scale invariance with parameters C_β and C_{ln}. Consider an interval of duration d that for $d \le D$ is entirely inside a storm and for $d > D$ includes one and only one storm. Then the rainfall intensity I_d satisfies,

$$I_d \overset{d}{=} \begin{cases} I_D' A_{Dr_Z/d}\,, & \text{for } d \le D \\ I_D' A_{r_Z} \dfrac{D}{d}\,, & \text{for } d > D \end{cases} \tag{5}$$

where the variables A_r have the distribution in equation (2). The expression for $d \le D$ is the product of the bare rainfall intensity at scale D, I_D', the scaling factor $A_{D/d}$ to obtain the bare intensity at scale d, and the factor A_{r_Z} that approximates the effect of dressing. As was explained in Section 2.2, the last two factors are combined into the term $A_{Dr_Z/d}$ The expression for $d > D$ in equation (5) has a similar structure and holds under the assumption that not more than one storm contributes rainfall in an interval of duration d. This assumption is accurate for small d (say $d \le 1$ day), but leads to some underestimation of the IDF values for d longer than a few days; see Section 3.2d.

The critical step in calculating the IDF values $i_{d,T}$ is to find the distribution of $M_{d|\boldsymbol{\theta}}$, the maximum of I_d in a storm with characteristics $\boldsymbol{\theta} = [I_D', D, C_\beta, C_{ln}]$. Following Veneziano and Langousis (2005b), for return periods T and resolutions D/d of practical interest one can neglect temporal dependencies and approximate the distribution of M_d as

$$F_{M_{d|\boldsymbol{\theta}}}(i) = \begin{cases} [F_{A_{Dr_Z/d}}(i/I_D')]^{D/d}, & \text{for } d \le D \\ F_{A_{r_Z}}(id/I_D'D), & \text{for } d > D \end{cases} \tag{6}$$

where F_X is the cumulative distribution function (CDF) of the random variable X. Then, the IDF values $i_{d,T}$ are given by

$$i_{d,T} = \begin{cases} I_D' \, A_{Dr_Z/d,(1-1/\lambda T)^{d/D}}, & \text{for } d \le D \\ I_D' \, \dfrac{D}{d} A_{r_Z,1-1/\lambda T}, & \text{for } d > D \end{cases} \tag{7}$$

where λ is the rate of storm arrivals and $A_{r,x}$ denotes the x-quantile of A_r. Equation (7) follows directly from equation (5).

Equation (7) applies when the storms have identical characteristics $\boldsymbol{\theta} = [I_D', D, C_\beta, C_{ln}]$ or storms with a given $\boldsymbol{\theta}$ dominate the IDF curves. When $\boldsymbol{\theta}$ varies randomly from storm to storm, the marginal distribution of M_d for a generic storm is obtained by taking expectation of equation (6) with respect to $\boldsymbol{\theta}$:

$$F_{M_d}(i) = \underset{\boldsymbol{\theta}}{E}[F_{M_{d|\boldsymbol{\theta}}}(i)] \tag{8}$$

The distribution function $F_{M_{d|\boldsymbol{\theta}}}$ is analytically known [see equation (6)] and calculation of F_{M_d} can proceed numerically without the use of Monte Carlo simulation; see also Sections 3.2d, 4.2, and 5. The IDF value $i_{d,T}$ is the $(1-1/\lambda T)$-quantile of M_d in equation (8).

3.2 Application to the Florence Rainfall Record

We exemplify the use of Model 1 by using a rainfall record from Florence, Italy (Becchi and Castelli, 1989); see Veneziano and Iacobellis (2002) for an analysis of this record. The data cover continuously the 24-year period from 1962 to 1985, at 5-min resolution. However, the rainfall intensity values at this maximum resolution are not reliable and the analysis that follows uses statistics and produces results only for durations of at least 20 minutes. In 1966, the City of Florence experienced a devastating flood. The unusually long and intense rainfall event associated with that flood is thought to have a return period far longer than the 24-year duration of the record. For this reason, in some of the analyses reported below, we have excluded the year 1966 and used only the remaining 23 years of the record.

(a) Rainstorm Identification

To extract storm characteristics from the data, one must first define what constitutes a rainstorm. One often identifies rainstorms as uninterrupted sequences of rainy periods. This is a strict criterion considering that even a very short pause in rainfall would split what meteorologically is a single storm into multiple events. To avoid these artificial splits, one must allow storms to include some dry intervals. A commonly used method is the so-called "dry period criterion", whereby dry intervals of duration longer than a given threshold τ_o are taken as storm separators. Selection of τ_o is either judgmental (e.g. Huff, 1967; Menabde and Sivapalan, 2000; Upton, 2002) or based on statistical tests of independence of the resulting storm events (Koutsoyiannis and Foufoula-Georgiou, 1993).

In all cases mentioned above, τ_o is taken to be a constant independent of storm duration. However, physical considerations suggest that τ_o should depend on storm type and size; see for example Orlanski (1975). Here we allow τ_o to scale with storm duration, as follows. We partition the record into continuously wet and continuously dry periods and then examine the dry periods in order of increasing duration. If a dry period is flanked by rainy intervals of the same or longer duration, the dry period is considered part of a storm interior (the dry period is reclassified as "wet", although it retains its zero rainfall intensity). The above procedure is applied with two exceptions: 1) dry periods of duration below some lower limit τ_o^- are always considered part of a storm interior, and 2) dry periods of duration longer than an upper limit τ_o^+ are always considered storm separators. This storm-identification procedure allows longer dry spells to be part of the interior of longer storms in a scale-invariant way. The values τ_o^- = 1 hr and τ_o^+ = 6 hr produce satisfactory results (see Section 3.2d) and are close to the maximum and minimum dry periods used in other studies (30 min in Upton, 2002; 1 hr in Menabde and Sivapalan, 2000; 6 hr in Huff, 1967, and 7 hr in Koutsoyiannis and Foufoula Georgiou, 1993). The impact of changing from variable to fixed τ_o is not large for fixed τ_o around 6 hr, but becomes noticeable if τ_o is taken to be 1 hr or less.

When applied to the 23 year Florence record, the variable τ_o approach with τ_o^- = 1 hr and τ_o^+ = 6 hr, identifies 844 storms with duration between 5 and 20 min and 3078 storms with duration between 20 and 2420 minutes.

(b) Joint Distribution of *D* and *I*

Figure 2a shows a scatterplot of the dressed intensity I_D and duration D for the 3078 storms with duration between 20 and 2420 minutes. A joint distribution was fitted to these data by estimating the marginal distribution of D and the conditional distribution of $(I_D|D)$ for different D intervals. Storms with duration smaller than 20 min were not used when fitting the joint distribution of D and I_D, because high-resolution values are not very accurate (see above) and small duration storms contribute relatively little to the IDF curves. As Figure 2b shows, storm duration D closely follows the 3-parameter Gamma distribution with probability density (for D in minutes)

$$f_D(x) = \frac{\delta^{\kappa}}{\Gamma(\kappa)}(x\text{-}c)^{\kappa\text{-}1}\mathrm{e}^{-\delta(x\text{-}c)}\,, \qquad \kappa, \delta > 0;\; x > c \tag{9}$$

and parameters δ = 0.00358 min^{-1}, κ = 0.613 and c = 5 min obtained by maximum likelihood.

To find the distribution of $(I_D|D)$, we binned the 3078 storms into 13 duration classes of equal log width between 20 and 2420 min and for each class fitted a normal distribution $\ln(I_D|D) \sim N[\mu(D), \sigma^2(D)]$ to the upper 25th percentile of the log data using maximum likelihood with censoring (Johnson and Kotz, 1970, p. 77). The empirical histograms of $\ln(I_D|D)$ do not generally conform to a normal model. However, as exemplified in Figure 2c for one duration interval, the upper tails of the $\ln(I_D|D)$ histograms are fitted well by a normal distribution. The upper quartile of the data has been selected because interest is in the upper tail and the data in that region are accurately fitted by normal distributions. Figure 2d shows the maximum-likelihood estimates of $\mu(D)$ and $\sigma(D)$ and the analytical fits

$$\begin{aligned} \mu(D) &= -0.052 \ln(D) + 0.377 \\ \sigma(D) &= 7.325\,\{\ln(D)\}^{-1.286} \end{aligned} \tag{10}$$

(c) Multifractal Parameters

When one analyzes continuous rainfall records without distinguishing between storms and inter-storm periods, one typically partitions the record into intervals of different duration d and examines how $\log E[(I_d)^q]$ varies with $\log(d)$ for different q; see for example Gupta and Waymire (1993), Olsson (1995), and Lovejoy and Schertzer (1995). Since our data consist of storms of different duration D and different average intensity I_D, the above procedure would produce biased results because only storms of duration $D \geq d$ can be used to obtain the scaling of the moment $E[(I_d)^q]$, and the distribution of storm intensity varies with D.

To avoid these problems we standardize each storm to have unit average intensity $\tilde{I}_D$, calculate the moments of the standardized rainfall intensities for different relative averaging durations $d' = d/D \leq 1$, and estimate $K(q)$ as the negative slope of $\log E[(\tilde{I}_{d'})^q]$ against $\log(d')$. The standardization step causes all the moments of $\tilde{I}_{d'}$ to be finite (by contrast, the moments of I_d of order $q \geq q^*$ diverge) and in particular affects the moments of $\tilde{I}_{d'}$ for d' close to 1. However, for small q and $d' << 1$ the bias in $K(q)$ is small.

Results (moment plot, empirical $K(q)$ function, and "beta-lognormal" fit) for the 3078 storms identified in the 23 year Florence record are shown in Figure 3. Lack of linearity of the moment plots for $d' > 0.5$ is due to normalization of the storm intensities, as mentioned above. For the estimation of $K(q)$, we have used the range $d' < 0.5$, where the moment plots are very nearly straight. For orders $q \leq 3$, the beta-lognormal fit to the empirical $K(q)$ function is quite good, whereas for $q > 3$ the empirical $K(q)$ becomes a nearly straight line. As noted already, this is due to a well-known bias when $K(q)$ is inferred from one-dimensional rainfall records (Ossiander and Waymire, 2000, 2002; Lashermes *et al.*, 2004; Veneziano *et al.*, 2006). The estimated multifractal parameters are $C_\beta = 0.025$ and $C_{ln} = 0.1$. Since the estimate of C_β is small, one may simplify the model by setting $C_\beta = 0$, which corresponds to a compact rainfall support inside storms.

To determine whether C_β and C_{ln} depend on storm duration and intensity, we have divided the storms into four categories, according to short/long duration and low/high average intensity. The threshold values of D and I_D were chosen to produce approximately the same number of storms in each class. The results are summarized in Table 1. Since the estimates of C_β and C_{ln} are nearly identical, we conclude that $C_\beta = 0$ and $C_{ln} = 0.1$ are appropriate values for all I_D and D.

(d) IDF Curves

In calculating the IDF curves $i_{d,T,}$ we assume that not more than one storm contributes rainfall in any given interval of duration d. This is equivalent to artificially spacing the storms by at least d. To assess the effects of this simplifying assumption, Figure 4 compares the empirical IDF curves from the 23 year Florence record with curves obtained by spacing the storms identified on that record by at least d. In all cases the Weibull (1939) plotting position formula is used; see for example Singh (1992). One can see that the assumption produces small distortions, except possibly for durations $d > 1$ day.

Calculation of the IDF values requires knowledge of the conditional distribution of $(I_D'|D)$. In Part b we have found the conditional distribution of the dressed intensity $(I_D|D)$. In analogy with equation (3), these distributions are related as $(I_D|D) \overset{d}{\approx} (I_D'|D)Z$. Hence, obtaining the distribution of $(I_D'|D)$ requires de-convolution of the distribution of the dressing factor Z from the distribution of $(I_D|D)$. This is in general a delicate numerically operation, but in our case it can be performed analytically because the distribution of $(I_D|D)$ was found to be approximately lognormal and, for $C_\beta = 0$, the distribution of A_{r_Z} that approximates Z is also lognormal; see equation (2). One concludes that $(I_D'|D)$ has lognormal distribution, with parameters

$$E[\ln I_D'|D] = E[\ln I_D|D] - E[\ln A_{r_Z}] \quad \text{(a)}$$
$$\mathrm{Var}[\ln I_D'|D] = \mathrm{Var}[\ln I_D|D] - \mathrm{Var}[\ln A_{r_Z}] \quad \text{(b)} \qquad (11)$$

where $E[\ln A_{r_Z}] = -C_{ln} \ln(r_Z)$, $\mathrm{Var}[\ln A_{r_Z}] = 2C_{ln} \ln(r_Z)$ and r_Z is given by equation (4). The results in Figure 5 were obtained by matching the 2nd moment of Z (equation 4a) Similar results are obtained when matching the 3rd moment of Z (equation 4b).

Under Model 1, the IDF curves are obtained using equation (8). When, as is the case here, the parameters (C_β, C_{ln}) do not vary with storm duration and average intensity, equation (8) reduces to,

$$F_{M_d}(i) = \int_D \int_{I_D'} F_{M_{d|I,D}}(i)\, f_{I_D'|D}(I)\, f_D(D)\; dI\, dD \tag{12}$$

where integration is over all possible durations D and bare average rainfall intensities I_D'. All distribution functions on the right hand side of equation (12) are known analytically (see above); hence evaluation of equation (12) can proceed numerically without the need for Monte Carlo simulation.

The IDF value $i_{d,T}$ is the $(1\text{-}1/\lambda T)$-quantile of M_d in equation (12). Note that, although the rainstorms used for model fitting have duration D of at least 20 minutes, the integration in equation (12) is performed over all theoretically possible durations D which, according to equation (9), extend down to c = 5min. Hence, the parameter λ should refer to all storms and its empirical value is λ=170.52 storms/year; see Section 3.2a.

Figure 5 compares the empirical IDF curves from the Florence record (solid lines) with curves from Model 1 (dashed lines). In Figures 5a and 5b, the empirical curves are estimated using 23-years of data (excluding the year 1966), whereas Figure 5c uses all 24 years. For averaging durations d longer than about one day, the empirical 24-year curve in Figure 5c gives very high rainfall intensities. This is due to the flood event of November, 1966. The return period of that event cannot be estimated empirically, but according to Model 1 it is on the order of 1000 years.

This is in general agreement with the historical flood record for the city of Florence (Becchi and Giuli, 1987).

The model-based IDF curves in Figures 5a, 5b and 5c are calculated by fitting the interior and exterior processes of Model 1 to 23 years (excluding 1966), the first 4 years (1962-1965) and all 24 years of the record, respectively. It is interesting that the results are nearly the same in all cases. This means that the analysis is insensitive to outlier storms and produces accurate results also from short records (we have obtained similar results using other 4-year segments of the data). The model curves in Figure 5a appear as smoothed versions of the empirical ones. The model curves slightly under-predict the intensities for long averaging durations, due to the assumption that no more than one storm contributes rainfall in any given d-interval; see above. For durations below 1 day, the bias from this assumption is negligible; see Figure 4.

Close inspection of the data explains the waviness of the empirical curves in Figure 5a. The relative highs around d = 100 and 1000 min are due to a few intense storms with those approximate durations (see highlighted events in Figure 2a), which introduce inflection points in the empirical IDF curves. The decrease in the slope of these curves for d smaller than 1 hour is due to the fact that at high resolutions the data display less variability than expected under exact multifractal scaling. This phenomenon has been observed also in other rainfall time series (Olsson *et al.*, 1993; Olsson 1995; Fraedrich and Larnder, 1993; Menabde and Sivapalan, 2000; Koutsoyiannis, 2006), and its effect on the IDF curves is usually accounted for in the parametric expressions for $i_{d,T}$ used in engineering practice (see for example Chow *et al.*, 1988; Singh, 1992; Koutsoyiannis *et al.*, 1998). The IDF results from Model 1 display some log-log curvature depending on the storm mixture that makes up the rainfall climate but do not include curvature from small-scale smoothness. At least in the case of the Florence record, some of the curvature of the empirical IDF plots for small-durations is due to artificial smoothing introduced by the digitization procedure (Castelli*,* personal communication).

Model 1 was used also to produce seasonal IDF curves (not shown here). This was done by dividing the 23-year Florence record into four seasonal sub-records, each for 3 calendar months. The empirical and model curves are always in good agreement, as in the case of annual analysis.

4. Parameter Estimation and IDF Analysis for Model 2

4.1 Estimation of Model Parameters

For Model 2, it is convenient to start by estimating the parameters C_β and C_{ln} of the interior process. Estimation is simpler than for Model 1, since one can use the continuous record to determine the scaling of the moments. In Figure 6a, we have divided the Florence rainfall intensities by the historical mean intensity $\hat{I}$ = 0.086 mm/hr and plotted the log moments $\log E[(I_d/\hat{I})^q]$ against $\log(d)$ for durations d up to 15 days. A good fit to the empirical $K(q)$ function in the range $0 \le q \le 3.5$ is obtained with $C_\beta = 0.5$ and $C_{ln} = 0.05$; see Figure 6b.

The estimates of C_β and C_{ln} are very different from Model 1. The much higher value of C_β is due to the predominance of dry conditions inside the D intervals of Model 2. The smaller value of C_{ln} in Model 2 is less easy to interpret, since in both models this parameter controls the variability of rainfall intensity when it rains. Hence C_{ln} should be the same. We have examined this issue in some detail and found that the discrepancy is due to the interstorm periods. Specifically, we have analyzed two synthetic rainfall series: a beta-lognormal series generated using the parameters $C_\beta = 0.5$ and $C_{ln} = 0.05$ of Model 2 and a series with storms of random duration and interior lognormal multifractality with $C_{ln} = 0.1$ as in Model 1. In the latter case, the storms are separated by dry periods of constant length. When the first series is analyzed the multifractal parameters are recovered, whereas for the second series one significantly underestimates C_{ln}. In essence, by assuming that lacunarity extends to the rainy periods, Model 2 interprets some of the variability inside storms as due to the beta component and consequently reduces the estimate of C_{ln}. In summary, the parameters obtained for Model 2 produce the best overall fit of a very simple model to a complicated process.

Next we turn to the parameters D and $F_{I_D'}$ of the exterior process. As was explained in Section 2.3, these parameters are used to reproduce the dressed moments of I_D of orders 0, 2 and 3. Using the approximation $Z \sim A_{r_Z}$ with r_Z from equation (4), the model moments $E[(I_D)^q]$ are the product of $E[(I_D')^q]$ times $E[(A_{r_Z})^q] = (r_Z)^{K(q)}$. Many ($D$, $F_{I_D'}$) pairs produce equally good moment fits, but it is desirable to use long durations D because Model 2 produces reliable IDF curves only for $d \leq D$. The 0th order moment $E[(I_D)^0]$ increases with increasing D but should be no less than the 0th order moment of the dressing factor, $E[(A_{r_Z})^0] = (r_Z)^{-C_\beta}$ (other positive contributions to $E[(I_D)^0]$ may come from the bare distribution of I_D', which may have a probability atom at zero). Hence D is maximized by finding the duration for which the empirical 0th order moment in Figure 6a equals this theoretical minimum. Using equation (4b), one finds $r_Z = 4.0$, $E[(A_{r_Z})^0] = 0.5$, and $D \approx 3.38$ days. For this value of D and the multifractal parameters estimated above, the empirical second and third moments of I_D are reproduced by taking $F_{I_D'}$ to be lognormal with parameters $\mu_{\ln I_D'}$= -2.570 and $(\sigma_{\ln I_D'})^2 = 0.475$. The solid lines in Figure 6a show that the moments from the fitted model agree very well with the empirical ones. For $d = D$, the theoretical moments in Figure 6a correspond to those of the lognormal exterior process. Equally good estimates are obtained when matching the 2nd moment of Z through equation (4a). This gives $r_Z = 3.35$, $E[(A_{r_Z})^0] = 0.45$ and $D = 4.18$ days.

As an alternative to matching moments, one can obtain $\mu_{\ln I_D'}$, and $(\sigma_{\ln I_D'})^2$ to fit the upper tail of $(I_D|I_D >0)$. We applied this procedure to the upper 25th of the distribution of $(I_D|I_D >0)$ using maximum likelihood with censoring. The resulting estimates are very similar to those from moment fitting.

4.2 IDF Curves

With the parameters given above, estimation of the IDF curves under Model 2 proceeds like for Model 1, with the simplification that the duration D is fixed. In this case equation (12) reduces to

$$F_{M_d}(i) = \int_{I_D'} F_{M_{d|I}}(i)\, f_{I_D'}(I)\, dI \tag{13}$$

The IDF value $i_{d,T}$ is obtained as the (1-1/λT)-quantile of M_d with $\lambda = 1/D$ (D in years, fixed).

Figure 6c compares the empirical IDF curves from 23 years of the Florence record with the IDF curves generated by Model 2. The agreement is again very good. As for Model 1, similar results were obtained when using all 24 years of the record or just 4-year segments. Comparing Figures 5a and 6c, one observes that the IDF curves from Model 2 are straightened versions of those of Model 1. Specifically, Model 2 captures less well the "hump" of the empirical IDFs around d = 100 min and the slope decrease for d < 1 hour. The reason is that these features are related, respectively, to the high intensity of short-duration storms and the lack of non-rainy periods at small scales, which are not resolved by Model 2. On the other hand, Model 2 uses rather large D values and suffers less from the assumption that not more than one D period contributes rainfall in d, for $d \le D$. This is why for large d the IDF curves from Model 2 are more accurate than those from Model 1.

5. Parameter Estimation and IDF Analysis for Model 3

Model 3 is a purely multifractal model in which the variability of I_D (the rainfall intensity in D) is due entirely to the dressing factor Z or its approximation A_{r_Z}. Hence, the bare density I_D'

reduces to a constant and can be estimated as the mean rainfall intensity of the entire record, $\hat{I}$ = 0.086 mm/hr. Inference of the interior multifractal parameters follows the same procedure as for Model 2, again producing the estimates $C_\beta = 0.5$ and $C_{ln} = 0.05$.

In the estimation of D, it is advisable to match a high empirical moment of I_D. Here we elect to reproduce the third moment, and therefore we calculate D from the condition $E[(I_D/\hat{I})^3] = E[Z^3] = r_Z^{\,K(3)}$, where $r_Z = 4.0$ from equation (4b). According to Figure 7a, the third moment of I_D is reproduced by taking $D = 14.6$ days. Estimation of $i_{d,T}$ follows directly from equation (7) and $\lambda = 1/D$ (D in years, fixed).

Figure 7a compares empirical and model moments. While the higher moments are well reproduced, there is a clear discrepancy in the 0^{th} order moment. This means that the wet/dry patterns generated by the model are different from the empirical ones. However, as one can see from Figure 7b, this discrepancy is not critical for extreme-value analysis. As for the other models, the results are insensitive to the presence of outliers and the length of the record used to fit the model.

6. Conclusions

We have developed parameter and IDF-curve estimation procedures for three rainfall models with local multifractal scale invariance. Each model consists of an interior multifractal process for the high-frequency fluctuations and an exterior process for the fluctuations at larger scales. By focusing on the characteristics or rainfall that are important for IDF analysis (the marginal distribution or marginal moments of the average rainfall intensity for different durations), the models are kept relative simple and the parameterization is parsimonious.

The most elaborate model (Model 1) uses the classical notion of rainstorms with random duration D and random average intensity I and represents the storm interior through a stationary multifractal process. In an application to a 24-year rainfall record from Florence, Italy, we have found that storm duration D has a three-parameter gamma distribution and the upper tail of the conditional storm intensity $(I|D)$ is lognormal with parameters that depend on D. Dry periods are allowed to occur within storms by including a "beta component" in the multifractal model, but the rainfall support inside the storms is nearly compact and a one-parameter lognormal multifractal model suffices. Importantly, we have found that the multifractal parameter can be taken to be the same for all D and I.

The model results (including extrapolation to longer return periods) are robust against outliers (we have included or excluded the year 1966 when the city of Florence experienced a record flood) and insensitive to the length of the rainfall record (similar IDF curves result from fitting the model to 4-year segments of the historical record). These features derive from the fact that the model makes efficient use of information from the continuous rainfall record, not just the annual extremes. This is an important advantage of the present rainfall modeling approach over the standard procedure of making parametric assumptions directly on the IDF curves and using the historical annual maxima.

Model 1 is attractive in that it uses the physical notion of rainstorm events. By considering the storm mixture that makes up the rainfall climate, this model is capable of producing curved IDF plots in log-log scale. This model is recommended in climates where particular storm types, for example convective storms associated with the diurnal cycle, have a dominant effect on the IDF curves for the return periods and averaging durations of interest. On the other hand, use of Model 1 is laborious because it requires the identification of storms, the estimation of the joint distribution of D and I, and careful analysis of the multifractality of the interior process.

Model 2 is much simpler, as it represents the rainfall process as a sequence of intervals of fixed duration D, each with random average intensity I. Inside each D-interval, rainfall intensity varies as a beta-lognormal multifractal process, in this case with a significant beta component. The beta component breaks down the D intervals into "rainstorms" of different durations and intensities, producing realistic rain patterns. The duration D is found as the outer limit of multifractal behavior and the distribution of I is estimated such that the upper tail or some higher moments of the marginal distribution of rainfall intensity are reproduced. Model 2 uses a very schematic representation of rainfall, but it too produces accurate and robust IDF curve estimates.

Model 3 is even simpler. It has a structure similar to Model 2, but both D and I are deterministic. Hence Model 3 represents rainfall as a series of multifractal cascades with no additional variability. Since the model does not consider rainfall variability at scales larger than D, this duration is estimated to be close to the storm interarrival time (for Florence, this is on the order of 15 days). By fitting the model parameters to the high moments, also Model 3 produces accurate and robust IDF results, with limited effort. For durations $d < D$, the IDF curves generated by Models 2 and 3 are nearly straight in log-log scale, because both models assume rainfall scaling below D. Therefore these models are applicable in the range of durations for which the moments of rainfall intensity have power-law dependence on d. In many moderate climates this range is between about 1 hour and a few days.

An important conclusion is that, more than the choice of model, what counts for IDF estimation are the criteria used for parameters estimation. When a moment method is used, the empirical rainfall moments to be reproduced should be those of order 2-3 for rainfall intensity in intervals of short relative duration d/D and order up to 4 for d/D close to 1. The reason for not using higher rainfall moments is that, especially for $d/D \ll 1$, those moments are underestimated with high probability. If additional parameters are available, as for example is the case for Model 2, then one could reproduce the probability of dry periods by matching also the 0^{th} order moment.

The simpler multifractal models, in particular Model 3, are amenable to analytical treatment and emerge as strong competitors of classical IDF estimation methods that fit parametric IDF models to rainfall annual maxima. The development of practical multifractal methods and comparison with standard approaches will be the subject of follow-up communications.

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References

Becchi, I. and Castelli, F. (1989) Lettura e archiviazione digitali di registrazioni pluviografiche ad alta risoluzione temporale, Technical Report 1/89, Dep. of Civil Eng., University of Firenze, Italy.

Becchi, I. and Giuli, D. (1987) Description of the Arno Project: A Real-time Approach to the Arno River Flooding Forecast, *Proceedings, International Conference on the Arno Project,* National Group for the Prevention of Hydro-geological Disasters, CNR, Tecnoprint, Bologna, pp. 9-49.

Bernard, M.M. (1932) Formulas for Rainfall Intensities of Long Durations, *Trans. ASCE,* **96**: 592-624.

Burlando, P., and Rosso, R. (1996), Scaling and Multiscaling Models of Depth-Duration-Frequency Curves for Storm Precipitation, *J. Hydrol.*, **187**: 45-64.

Chow, V.T., Maidment, D.R., and Mays, L.W. (1988) *Applied Hydrology*, McGraw-Hill, New York.

Cowpertwait, P.S.P. (1995) A Generalized Spatial-Temporal Model of Rainfall Based on a Clustered Point Process, *Proc. Royal Soc. London, Ser. A,* **450**: 163-175.

Cowpertwait, P.S.P. (1998) A Poisson-Cluster Model of Rainfall: High-Order Moments and Extreme Values, *Proc. Royal Soc. London, Ser. A,* **454**: 885-898.

Deidda, R. (2000) Rainfall Downscaling in a Space-Time Multifractal Framework, *Wat. Resour. Res.*, **36**(7), 1779-1794.

Eagleson, P.S. (1970), *Dynamic Hydrology*, McGraw-Hill, New York.

Fraedrich, K. and Larnder, C. (1993) Scaling Regimes of Composite Rainfall Time Series, *Tellus*, **45A**, 289-298.

Gupta, V.K. and Waymire, E.C. (1993) A Statistical Analysis of Mesoscale Rainfall as a Random Cascade, *J. Appl. Meteor.*, **32**, 251-267.

Harris, D., Menabde, M., Seed, A., and Austin, G. (1998) Breakdown Coefficients and Scaling Properties of Rain Fields. *Nonlinear Processes in Geophys.*, **5**, 93-104.

Hubert, P., Bendjoudi, H., Schertzer, D., and Lovejoy, S. (1998) A Multifractal Explanation for Rainfall Intensity-Duration-Frequency Curves, *Proceedings,* Int. Conf. On Heavy Rains and Flash Floods, National Research Council, Group for prevention from Hydrological Disasters, Istanbul, Turkey.

Huff, F.A. (1967) Time Distribution of Rainfall in Heavy Storms, *Wat. Resour. Res.*, **3**(4), 1007-1019.

Johnson, N. L. and Kotz, S. (1970), *Distributions in Statistics: Continuous Univariate Distributions – I*, Wiley, New York.

Kahane, J.P., and Peyriere, J. (1976) Sur certaines martingales de Benoit Mandelbrot, *Adv. Math.,* **22**: 131-145.

Koutsoyiannis, D. (2004a) Statistics of Extremes and Estimation of Extreme Rainfall, 1,Theoretical Investigation, *Hydrological Sciences Journal*, **49**(4), 575-590.

Koutsoyiannis, D. (2004b) Statistics of Extremes and Estimation of Extreme Rainfall, 2, Empirical Investigation of Long Rainfall Records, *Hydrological Sciences Journal*, **49**(4), 591-610.

Koutsoyiannis, D. (2006) An Entropic-Stochastic Representation of Rainfall Intermittency: The Origin of Clustering and Persistence, *Wat. Resour. Res.*, **42**(1), W01401.

Koutsoyiannis, D., and Foufoula-Georgiou, E. (1993) A Scaling Model for a Storm Hyetograph, *Wat. Resour. Res.*, **29**(7), 2345-2361.

Koutsoyiannis, D., Kozonis, D., and Manetas, A. (1998) A Mathematical Framework for Studying Rainfall Intensity-Duration-Frequency Relationships, *J. Hydrol.,* **206**: 118-135.

Lashermes, B., Abry, P., and Chainais, P. (2004), New Insights into the Estimation of Scaling Exponents, *Int. Jour. of Wavelets*, *Multir. and Inf. Proc.*, **2**, 497-523.

Lovejoy, S., and Schertzer, D. (1995) Multifractals and Rain, In: *New Uncertainty Concepts in Hydrology and Hydrological Modelling*, Edited by: Kundzewicz, A.W., Cambridge press, 62-103.

Menabde, M. and Sivapalan, M. (2000) Modeling of Rainfall Time Series and Extremes Using Bounded Random Cascades and Levy-stable Distributions, *Wat. Resour. Res.*, **36**(11): 3293-3300.

Menabde, M., Harris, D., Seed, A., Austin, G. and Stow, D. (1997) Multiscaling Properties of Rainfall and Bounded Random Cascades, *Wat. Resour. Res.*, **33**(12), 2823-2830.

Olsson, J. (1995) Limits and Characteristics of the Multifractal Behavior of a High-Resolution Rainfall Time Series, *Nonlinear processes in Geoph.*, **2**, 23-29.

Olsson, J., Niemczynowicz, J., and Berndtsson, R. (1993) Fractal Analysis of High Resolution Time Series, *J. Geophys. Res.*, **98**, 23265-23274.

Orlanski, I. (1975) A Rational Subdivision of Scales for Atmospheric Processes. *Amer. Meteor. Soc.,* **56**(5), 527-530.

Ossiander, M. and Waymire, E.C. (2000), Statistical Estimation for Multiplicative Cascades, *Ann. Statist.*, **28**, 1533-1560.

Ossiander, M. and Waymire, E.C. (2002), On Estimation Theory for Multiplicative Cascades, *The Indian Journal of Statistics*, **64**, 323-343

Over, T.M., and Gupta, V.K. (1996) A Space-Time Theory of Mesoscale Rainfal Using Random Cascades, *J. Geophys. Res.*, **101**(D21): 26 319- 26 331.

Schertzer, D. and Lovejoy, S. (1987) Physical Modeling and Analysis of Rain and Clouds by Anisotropic Scaling of Multiplicative Processes, *J. Geophys. Res.*, **92**: 9693-9714.

Schertzer, D. and Lovejoy, S. (1989) Generalized Scale Invariance and Multiplicative Processes in the Atmosphere, *Pageoph.*, **130**, 57-81.

Schmitt, F., Vannistem, S. and Barbosa, A. (1998) Modeling of Rainfall Time Series Using Two-state Renewal Processes and Multifractals, *J. Geophys. Res.*, **103** (*D18*): 23,181-23,193.

Sherman, C.W. (1931) Frequency and Intensity of Excessive Rainfall at Boston, *Trans. ASCE,* **95**:951-960.

Singh, V. P. (1992) *Elementary Hydrology*, Prentice-Hall, New Jersey, U.S.A.

Upton, G.J.J. (2002) A Correlation-Regression Method for Tracking Rainstorms Using Rain-Gauge Data. *J. Hydrol.*, **261**, 60-73.

Veneziano, D. and Furcolo, P. (2002) Multifractality of Rainfall and Intensity-duration-frequency Curves, *Wat. Resour. Res.*, **38**(12): 1306-1317.

Veneziano, D. and Furcolo, P. (2003) Marginal Distribution of Stationary Multifractal Measures and Their Haar Wavelet Coefficients, *Fractals*, **11**(3): 253-270.

Veneziano, D. and Iacobellis, V. (2002) Multiscaling pulse representation of temporal rainfall, *Wat. Resour. Res.*, **38**(8), 13.1-13.13.

Veneziano, D. and Langousis A. (2005a) The Areal Reduction Factor a Multifractal Analysis, *Wat. Resour. Res.*, **41**, W07008, doi:10.1029/2004WR003765.

Veneziano, D. and Langousis, A. (2005b) The Maximum of Multifractal Cascades: Exact Distribution and Approximations, *Fractals*, **13**(*4*): 1-14.

Veneziano, D., Langousis, A., and Furcolo, P. (2006) Multifractality and Rainfall Extremes: A Review, *Wat. Resour. Res.*, **42**, W06D15, doi:10.1029/2005WR004716.

Venugopal, V., Roux, S.G., Foufoula-Georgiou, E., and Arneodo, A. (2006) Revisiting Multifractality of High Resolution Temporal Rainfall Using a Wavelet-Based Formalism, *Wat. Resour. Res*., **42**, W06D14, doi:10.1029/2005WR004489.

Waymire, E.C. and Gupta, V.K. (1981) The Mathematical Structure of Rainfall Representation. 1. A Review of the Stochastic Rainfall Models, 2. A Review of the Point Processes Theory, 3. Some Applications of the Point Process Theory to Rainfall Processes, *Wat. Resour. Res.,* **17**(5): 1261-1294.

Weibull, W. (1939) A Statistical Theory of the Strength of Materials, *Ing. Vetenskaps Akad. Handl*, Stockholm, **151-3**: 45-55.

Willems, P. (2000) Compound Intensity/Duration/Frequency Relationships of Extreme Precipitation for Two Seasons and Two Storm Types, *J. Hydrol.,* **233**: 189-205.

Figure Captions

Figure 1: Schematic representation of the exterior process for rainfall models 1, 2, and 3.

Figure 2: Fitting of Model 1 to 23 years of the Florence rainfall record: (a) Scatter plot of the density I_D and duration D of the 3078 storms with duration larger than 20 min, (b) fitted gamma distribution of storm duration D, (c) example tail fitting of $(\ln I_D|D)$ for D between 35 and 55 minutes, and (d) mean and standard deviation of $(\ln I_D|D)$.

Figure 3: Estimation of C_β and C_{ln} for Model 1: (a) Moments of the normalized rainfall intensity as a function of the dimensionless duration $d'=d/D$. (b) Comparison of the empirical $K(q)$ function with the fitted $K(q)$ function for $C_{ln} = 0.1$ and $C_\beta = 0.025$.

Figure 4: Comparison of empirical IDF curves estimated from the 23-year continuous rainfall record (solid lines) with those obtained by extracting storms and assuming that not more than one storm contributes rainfall in each d- interval (dashed lines). Return periods T= 2, 4, 8, 24 years, increasing from below.

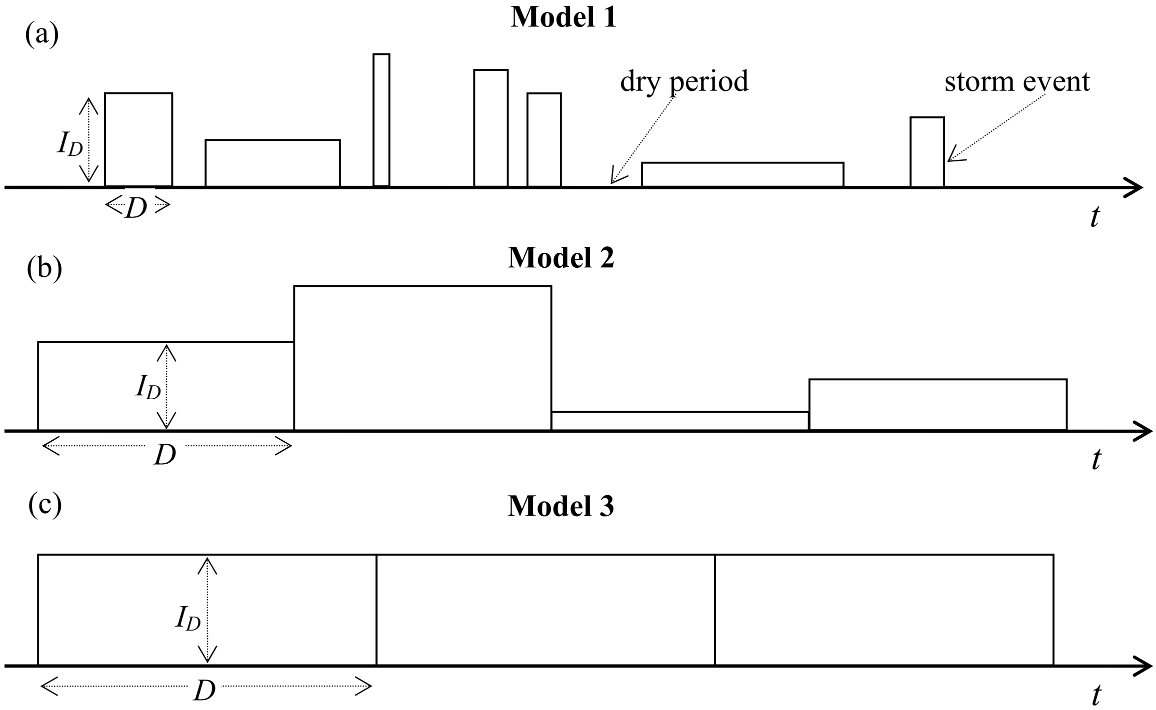
Figure 5: Comparison of empirical and theoretical IDF curves using Model 1. In (a) and (b) the empirical IDF curves are estimated from 23 years of the record, whereas in (c) estimation is from all 24 years. In Panels (a), (b) and (c) the model curves are obtained by fitting Model 1 to 23, 4 (1962-1965), and 24 years of the record, respectively.

Figure 6: Application of Model 2 to the Florence rainfall record, normalized to have unit average rain rate. (a) Moments of the 23-year rainfall record (circles) and moments from the fitted model (solid lines). (b) Comparison of the empirical and model moment scaling functions. (c) Comparison of empirical and model IDF curves.

Figure 7: Application of Model 3 to the Florence rainfall record. (a) Moments of the 23-year rainfall record (circles) and moments from the fitted model (solid lines). (b) Comparison of empirical and model IDF curves.

Table 1: Classification of the 3078 storms with duration $D \geq 20$ min into four sub-sets according to short/long duration D and low/high average intensity I_D. Multifractal parameters are estimated by the procedure of Section 3.2c.

Category	***Storm Duration (D)***	***Storm Intensity (I_D)***	***Number of storms***	C_β	C_{ln}
1	20min-3hrs	0.036-1.26 mm/hr	875	0.026	0.093
2	20min-3hrs	1.26-42 mm/hr	1038	0.018	0.108
3	3-40hrs	0.036-1.26 mm/hr	734	0.033	0.101
4	3-40hrs	1.26-42 mm/hr	431	0.023	0.097



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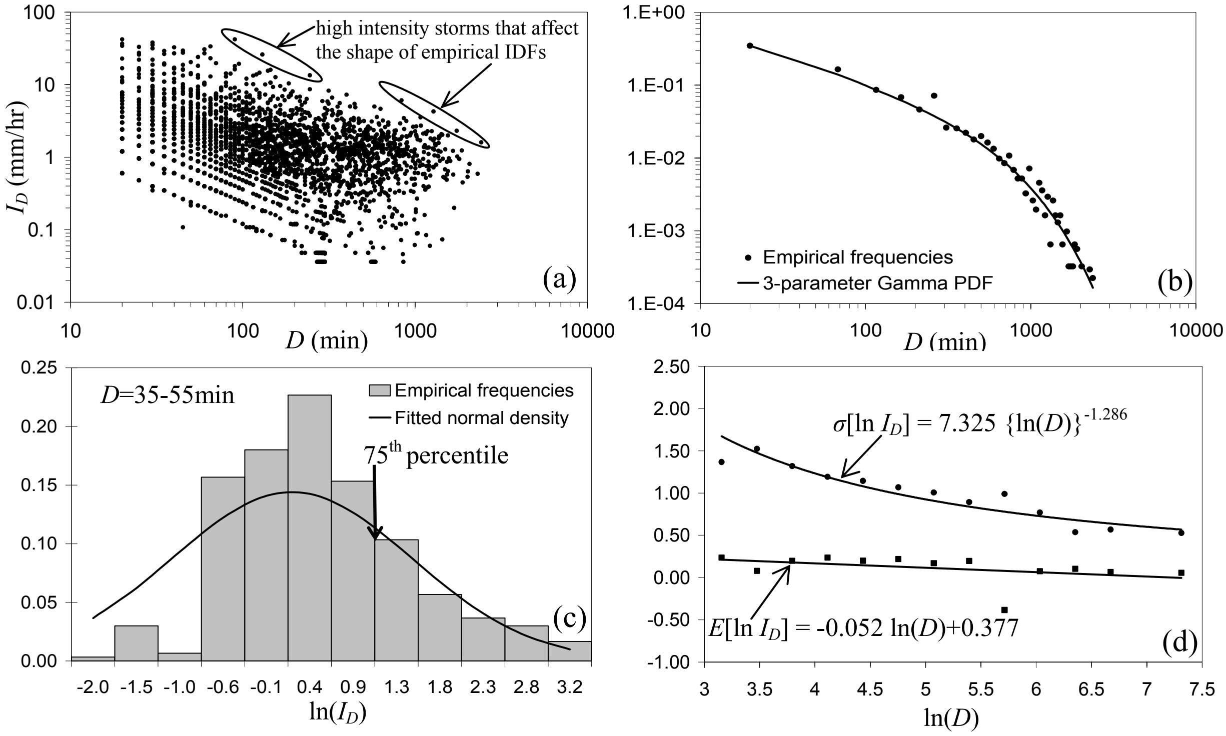

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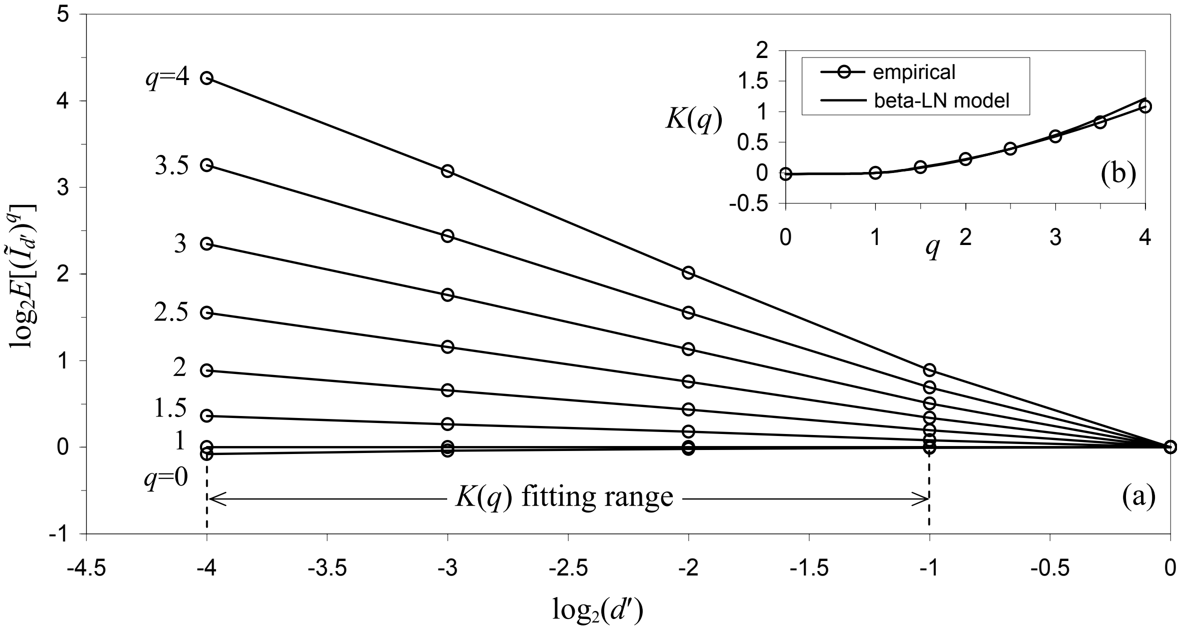

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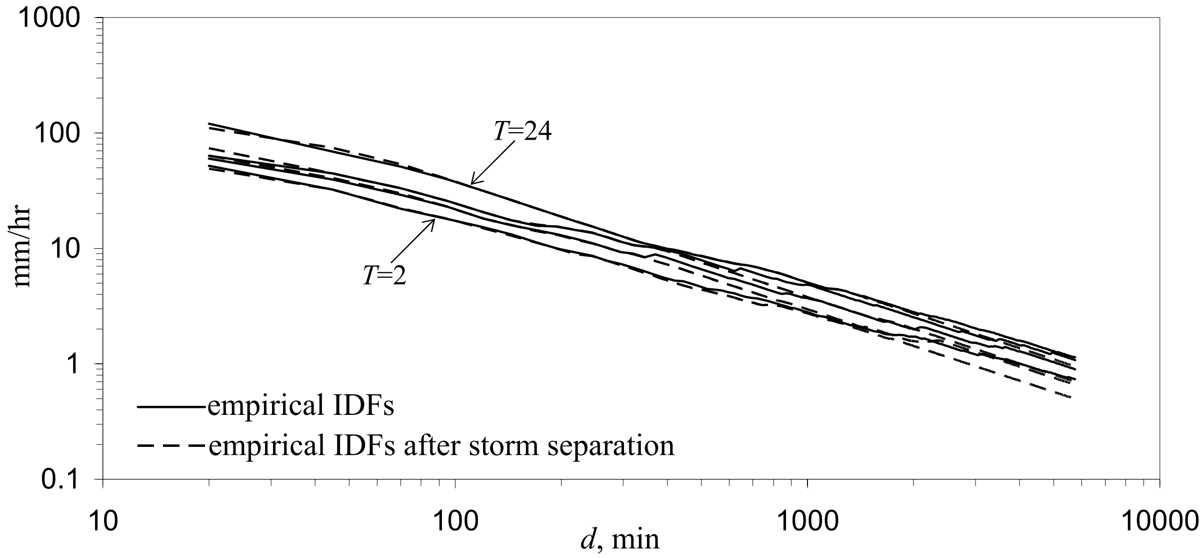

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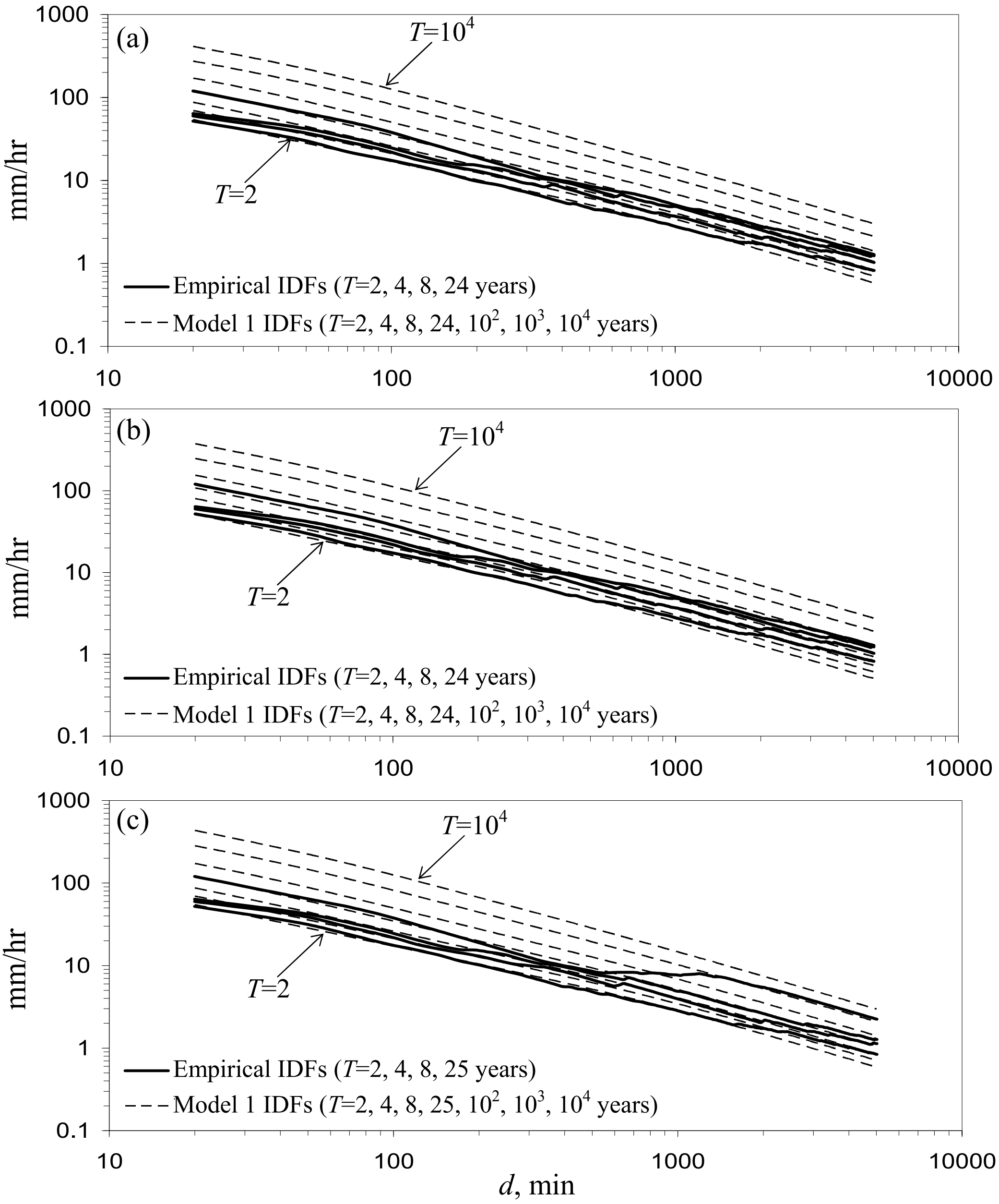

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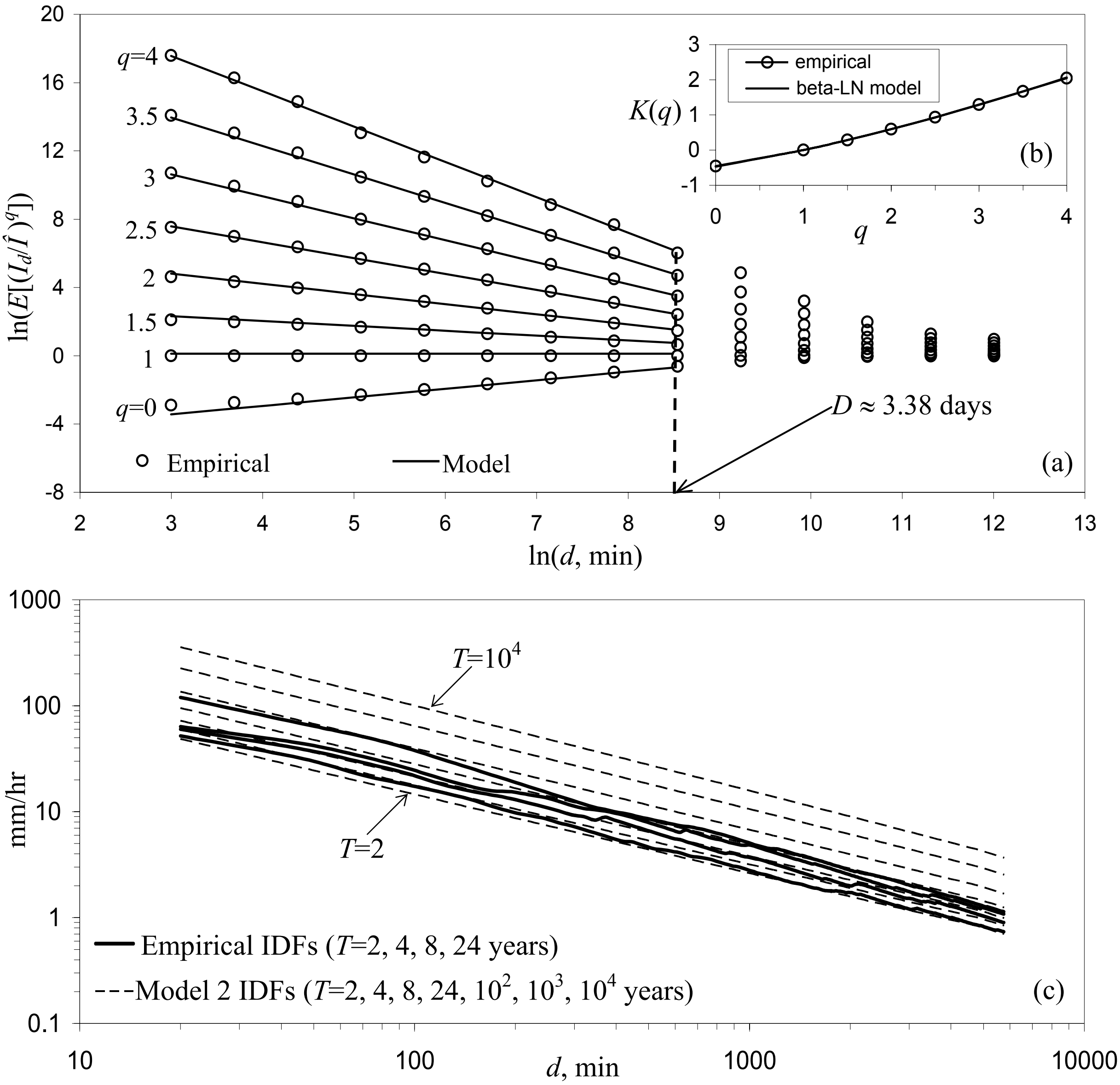

Figure 6: Application of Model 2 to the Florence rainfall record, normalized to have unit average rain rate. (a) Moments of the 23-year rainfall record (circles) and moments from the fitted model (solid lines). (b) Comparison of the empirical and model moment scaling functions. (c) Comparison of empirical and model IDF curves.

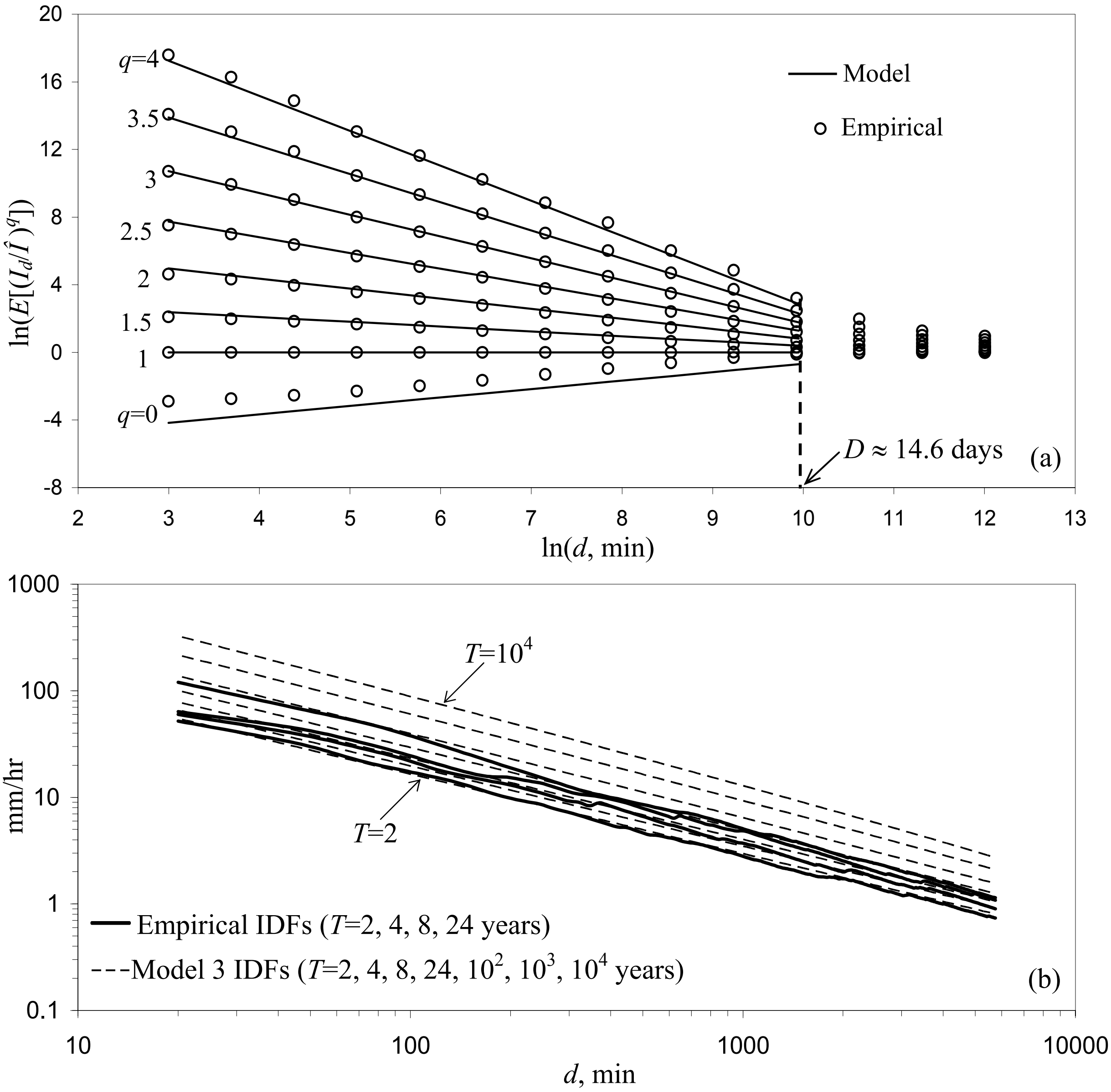

Figure 7: Application of Model 3 to the Florence rainfall record. (a) Moments of the 23-year rainfall record (circles) and moments from the fitted model (solid lines). (b) Comparison of empirical and model IDF curves.