European Geosciences Union, General Assembly 2005, Vienna Austria, 24-29 April 2005

The maximum of Multifractal Cascades: Exact Distribution and Approximations

Daniele Veneziano and Andreas Langousis
(venezian@mit.edu) (andlag@mit.edu)
Department of Civil and Environmental Engineering, M.I.T., Cambridge Mass., U.S.A.

Objective

Find the exact distribution of the maximum of a multifractal cascade and develop simple and accurate approximations.

1. Bare cascade construction and dressing

- Start at level 0 with a single d-dimensional cubic tile S and a unit measure with uniform bare density in S.
- At each subsequent level n = 1, 2, …, each tile at the previous level n - 1 is partitioned into $m = \mu^d$ cubic tiles S_i (i=1,…, m) where μ is an integer larger than 1; see **Figure 1**. The bare density in S_i is obtained as the product of the bare density in the parent tile at level n-1 and an independent copy Y_i of a non-negative, unit-mean random variable Y, called the cascade generator.

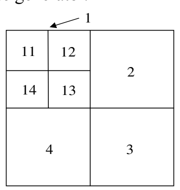

Figure 1: Illustration of a 2D binary cascade construction for m = 4

- We call m the *volumetric multiplicity* of the cascade and $r = m^n$ the *volumetric resolution* when the cascade has reached level n.

Dressing

$\varepsilon_{b,n}$ = bare measure density in a cascade tile at level n.
$\varepsilon_{d,n}$ = dressed measure density in a cascade tile at level n.
Z = dressing factor (Z has the same distribution as $\varepsilon_{d,0}$).
q^* = order above which the moments $E[Z^q]$ and $E[(\varepsilon_{d,n})^q]$ diverge.

<u>The bare and dressed densities satisfy:</u>

$$\begin{cases} \varepsilon_{b,n} \overset{d}{=} \prod_{i=1}^{n} Y_i \\ \varepsilon_{d,n} \overset{d}{=} \varepsilon_{b,n} Z \end{cases} \quad \text{For large } z \text{ and } \varepsilon: \quad \begin{cases} P[Z > z] \sim z^{-q^*} \\ P[\varepsilon_{d,n} > \varepsilon] \sim \varepsilon^{-q^*} \end{cases} \qquad (1)$$

2. Exact distribution of cascade maxima

Let $M_n = \max_{i=1,\dots,m^n}(\varepsilon_{d,n_i})$ be the maximum in S of the dressed measure density at resolution $r = m^n$. Its exact distribution F_{M_n} can be found through the following recursive procedure.

$$F_{\log M_0} = F_{\log Z} \quad , n = 0$$
$$F_{\log M_n} = (F_{\log M_{n-1}} * f_{\log Y})^m \; , n = 1, 2, \dots \qquad (2)$$

Where F_X and f_X denote the CDF and PDF of a random variable X. Notice that calculation of both F_{logZ} in the first step and $F_{log\,M_n}$ at each subsequent step requires numerical convolution.

➢ <u>*Hence need for approximations.*</u>

3. Effect of the multiplicity m

For a given bare moment scaling function $K(q) = \log_m[Y^q]$, does the maximum at resolution r, M_r, depend on the volumetric multiplicity m? If it does not, then:

- Discrete cascade extremes approximate well the extremes of continuous processes with the same $K(q)$;
- The dimensionality d of the support does not matter.

To answer this question, let $M_{r,T,m}$ be the upper (1/T)-quantile of M_r for a cascade with multiplicity m. For a specific lognormal cascade, **Figure 2** shows the ratio $M_{r,T,m=4}/M_{r,T,m=2}$ against r for different T.

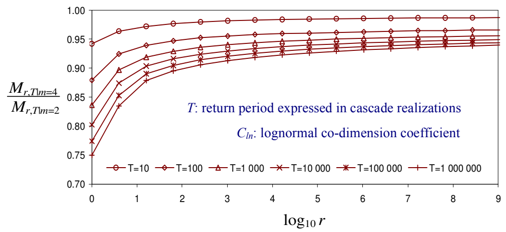

Figure 2: Ratio $M_{r,T,m=4}/M_{r,T,m=2}$ for a log normal cascade with C_{ln}=0.1.

From **Figure 2** and similar results for other beta-lognormal cascades one concludes that, as m increases:

- the correlation of the measure density between tiles at given distance decreases, producing higher values of $M_{r,T|m}$. However this is a *small effect*.
- the distribution of the dressing factor Z tightens around 1, causing $M_{r,T|m}$ to decrease. This is a *larger effect,* especially for small r and large T.

➢ <u>*Net effect:*</u> *$M_{r,T|m}$ decreases with increasing m.*

4. Approximations to M_n

We explore various approximations to M_n, in which the distribution of Z is simplified and the dependence among the measure densities in different cascade tiles is ignored or treated in approximation. An important consequence of these approximations is that <u>convolution operations are avoided</u>. Below we consider, in sequence, the effect of the following simplifications:

- *Assume tile independence.* As a part of this analysis, we study the separate effect of short-range and long-range dependences.
- *Approximate Z through a random variable with the distribution type of the bare density ε_b.*
- *Approximate Z and include tile dependence through a simplified correction factor on the quantiles of M_n.*

4.1 Effect of tile dependence

For a lognormal cascade with m = 2 and co-dimension parameter C_{ln}= 0.1, **Figure 3** compares the distributions of $M_{n=10}$ and $M_{n=20}$ with the same distributions under independence. The distribution under independence is found from:

$$F_{\log M_{n,ind}} = (F_{\log Z} * f_{\log Y_1+\dots+\log Y_n})^{m^n} \qquad (3)$$

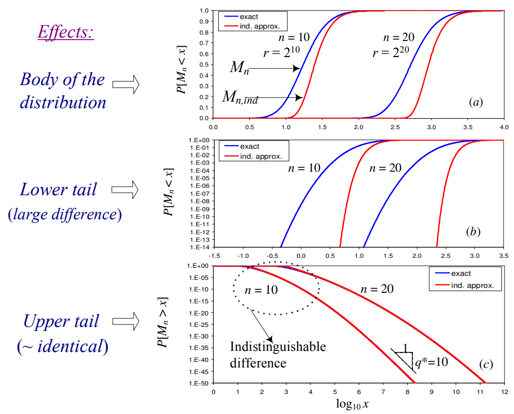

Figure 3: The effect of dependence among cascade tiles on the distribution of M_n.

- *Dependence among the cascade tiles affects the body and lower tail of the distribution of M_n but not the upper tail.*
- Having found that dependence among the cascade tiles has a significant effect on M_n, we investigate whether such effect comes mainly from *short-range* or *long-range* dependence. **Figure 4** compares the distribution of $M_{n=20}$ for the cascade in **Figure 3** when long-range or short-range dependence is progressively neglected. <u>We conclude that most of the effect comes from long-range dependence.</u>

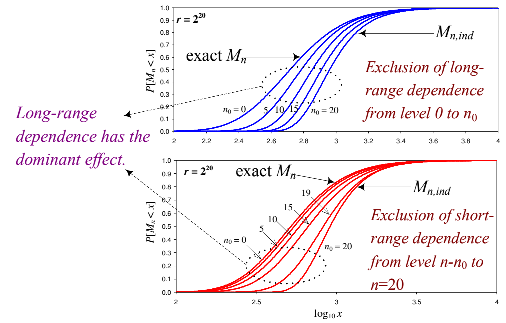

Figure 4: Change in the distribution of $M_{n=20}$ when long-range or short-range dependence is progressively excluded.

4.2 Approximation of Z

While possible, numerical calculation of the distribution of Z is tedious (Veneziano and Furcolo, 2003). Here we approximate Z with the bare density $\varepsilon_{b,\,r_Z}$ where the resolution r_Z is chosen to match some characteristic of Z. For example, if Z has finite second moment, then one may choose r_Z so that $E[(\varepsilon_{b,r_Z})^2] = r_Z^{K(2)}$ equals $E[Z^2] = \frac{m-1}{m - m^{K(2)}}$, giving:

$$r_Z = \left(\frac{m-1}{m - m^{K(2)}}\right)^{1/K(2)} \qquad (4)$$

Figure 5 compares the upper tail of the exact distribution of M_n with the distribution obtained with $\varepsilon_{b,\,r_Z}$ in place of Z. *The distributions are very similar* except for the combination of very low n and very high quantiles.

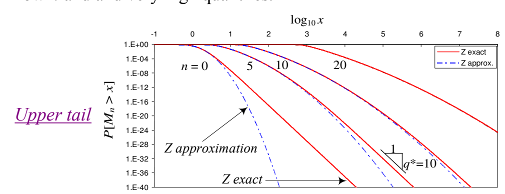

Figure 5: Lognormal cascade with C_{ln}=0.1. Comparison of the exact distribution of M_n with the distribution obtained by replacing Z with $\varepsilon_{b,\,r_Z}$

4.3 Proposed approximation for M_n

➢ Z can be accurately approximated by $\varepsilon_{b,\,r_Z}$, but tile dependence should not be ignored.

- To approximate the effect of dependence we express the upper (1/T)-quantile of M_n, $M_{n,T}$, as:

$$M_{n,T} \approx \gamma(n,T)\, M_{n,ind,T} \qquad (5)$$

where $\gamma(n,T)$ is a numerically evaluated correction factor. For the case of *beta-log normal cascades* we have found that $\gamma(n,T)$ depends minimally on the beta component.

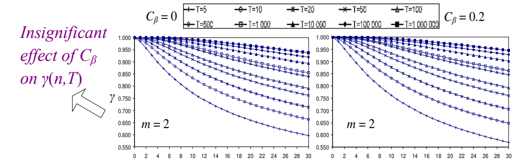

Figure 6: $\gamma(n, T)$ as a function of T and n. Lognormal and beta-lognormal cascades with parameters C_{ln} = 0.1 and C_β = 0, 0.2.

- We have also found that $\gamma(n,T)$ depends analytically on the lognormal co-dimension parameter C_{ln} as:

$$\gamma_{C_{ln_1}}(n,T) \approx \left(\gamma_{C_{ln_2}}(n,T)\right)^{\sqrt{C_{ln_1}/C_{ln_2}}}$$

- This is a special case of the dependence of $\gamma(n, T)$ on the C parameter of log-stable cascades; see Veneziano and Langousis (2005). Insensitivity to the beta component and the above dependence on C_{ln} make **Figure 6** sufficient to determine γ for all beta-lognormal cascades.

5. Conclusions

- While possible, numerical calculation of the exact distribution of the cascade maxima M_n is tedious;
- The cascade multiplicity m has a mild effect on the distribution of M_n at given volumetric resolution $r = m^n$. Therefore, the distribution of the maximum at resolution r depends little on the dimension d of the support;
- Dependence among the cascade tiles has an important effect on the body and lower tail of the distribution of M_n;
- For beta-log normal cascades, a simple and accurate approximation to M_n is obtained by combining Eqs. (3), (4) and (5) as follows:
  1. Calculate the parameter r_Z in the approximation of Z;
  2. Calculate the distribution of $M_{n,ind}$, the maximum for independent cascade tiles;
  3. Multiply the upper (1/T)-quantile of $M_{n,ind}$ by $\gamma(n,T)$.

Acknowledgements

This work was supported in part by the National Science Foundation under Grant No. EAR-0228835, and in part by the Alexander S. Onassis Public Benefit Foundation under Scholarship No. F-ZA 054/2004-2005.

References

Veneziano, D. and P. Furcolo (2003), "Marginal Distribution of Stationary Multifractal Measures and Their Haar Wavelet Coefficients," *Fractals,* **11**(3): 253-270.
Veneziano D. and Langousis A. (2005) "The maximum of Multifractal cascades exact distribution and approximations" (Submitted to Fractals).