Rainfall downscaling in time: theoretical and empirical comparison between multifractal and Hurst-Kolmogorov discrete random cascades

Descente d'échelle temporelle de précipitations: Comparaison théorique et empirique entre cascades discrètes aléatoires multifractales et Hurst-Kolmogorov

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Abstract During recent decades, intensive research has focused on techniques capable of generating rainfall time series at a fine time scale which are (fully or partially) consistent with a given series at a coarser time scale. Here we theoretically investigate the consequences on the ensemble statistical behaviour caused by the structure of a simple and widely-used approach of stochastic downscaling for rainfall time series, the discrete Multiplicative Random Cascade. We show that synthetic rainfall time series generated by these cascade models correspond to a stochastic process which is non-stationary, because its temporal autocorrelation structure depends on the position in time in an undesirable manner. Then, we propose and theoretically analyse an alternative downscaling approach based on the Hurst-Kolmogorov process, which is equally simple but is stationary. Finally, we provide Monte Carlo experiments that validate our theoretical results.

Key words Rainfall downscaling; Hurst-Kolmogorov process; multifractals; discrete random cascades; ensemble statistical behaviour; stationarity

Résumé Au cours des dernières décennies, une recherche intensive a mis l'accent sur des techniques capables de produire des séries chronologiques de précipitations à une échelle temporelle fine, qui sont (complètement ou partiellement) cohérentes avec une série donnée à une échelle temporelle plus grossière. Dans le présent article, nous étudions théoriquement les conséquences sur les tendances des statistiques d'ensemble causées par la structure d'une approche simple et largement utilisée de descente d'échelle stochastique pour séries temporelles de précipitations: La Cascade Aléatoire Multiplicative discrète. Nous démontrons que les séries temporelles synthétiques de précipitations, produites par le modèle de Cascade Aléatoire Multiplicative, correspondent à un processus stochastique qui n'est pas stationnaire, étant donné que son autocorrélation temporelle varie dans le temps de façon indésirable. Ensuite, nous présentons et analysons théoriquement une approche alternative de descente d'échelle fondée sur le processus de Hurst-Kolmogorov, qui est également simple, mais est stationnaire. Enfin, nous prouvons le bien-fondé de nos résultats théoriques avec la méthode de simulation de Monte-Carlo.

Mots clefs Descente d'échelle de précipitations; processus de Hurst-Kolmogorov; multifractales; cascades aléatoires discrètes; tendances des statistiques d'ensemble; stationnarité

1 INTRODUCTION

In stochastic hydrology we often need to study natural processes at different time scales. The problems associated with the transfer of information across scales have been called scale issues (Blöschl and Sivapalan 1995). To adequately address scale issues, we require models capable of preserving consistency across scales, i.e. both in a coarser, or higher-level, time scale and in a finer, or lower-level, time scale. These issues may arise, for instance, when coupling stochastic models of different time scales to reproduce simultaneously different important statistical properties of a hydrological process (Koutsoyiannis 2001), e.g. the long-term and the short-term stochastic structure of precipitation (Langousis and Koutsoyiannis 2006).

In other cases, scale issues are encountered in predictions using hydrological models, where the modelling scale may be much smaller than the observation scale; hence, we need to bridge that gap to calibrate, validate and operationally use our models. For example, when the higher-level process is the output of weather prediction models, which is given at a coarse scale, the scale discrepancy between model output and the resolution required for hydrological modelling must be resolved (e.g. Fowler *et al.* 2007, Groppelli *et al.* 2011). Furthermore, the higher-level process may be known from measurements. Specifically, when dealing with rainfall, long historical records usually come from daily rain gauges, but we need hourly or sub-hourly precipitation data in many hydrological applications. Also, the satellite rainfall data are available at a spatial scale greater than about 30 km at the Equator, and a temporal scale of 3 h, while again hydrological applications (e.g. related to flash floods) require higher resolutions (Berne *et al.* 2004, Koutsoyiannis and Langousis 2011).

Scale issues can potentially be tackled by both disaggregation and downscaling techniques, which aim at modelling linkages across different temporal and/or spatial scales of a given process. In stochastic hydrology, a natural process $R(t)$, e.g. rainfall, is usually defined at continuous time t, but we observe or study it at discrete time as $R_j^{(\delta)}$, which is the average of $R(t)$ over a fixed time scale δ at discrete time steps j (=1, 2, …), i.e.:

$$R_j^{(\delta)} := \frac{1}{\delta}\int_{(j-1)\delta}^{j\delta} R(t)\mathrm{d}t \quad (1)$$

Let $f\delta$ be a time scale larger than δ where f is a positive integer; for convenience δ will be omitted. Then, we can define the aggregated stochastic processes on that time scale, $Z_j^{(f)}$ (and relate it to the mean aggregated $R_j^{(f)}$) as:

$$Z_j^{(f)} := \sum_{l=(j-1)f+1}^{jf} R_l = fR_j^{(f)} \quad (2)$$

e.g. $Z_1^{(f)} = R_1 + \ldots + R_f$ and $Z_2^{(f)} = R_{f+1} + \ldots + R_{2f}$.

Both disaggregation and downscaling refer to transferring information from a given scale (higher-level) to a smaller scale (lower-level), e.g. they generate consistent rainfall time series at a specific scale given a known precipitation measured or simulated at a certain coarser scale. The two approaches are very similar in nature but not identical to each other. Downscaling aims at producing the finer-scale rain field with the required statistics, being statistically consistent with the given field at the coarser scale, while disaggregation has the additional requirement to produce a finer scale rain field that adds up to the given coarse-scale total; thus, in this case we introduce an equality constraint to the problem in the form of equation (2). The reader is referred to Koutsoyiannis and Langousis (2011) and the references therein for a detailed review on disaggregation and downscaling models in the literature.

This paper focuses on the analysis of discrete random cascades for rainfall downscaling, which are characterized by a very simple structure, easy to implement and, consequently, widely applied in the literature. Hence, we compare the ensemble behaviour of two simple rainfall downscaling models based on two similar approaches: the multifractal and the Hurst-Kolmogorov. Both approaches are based on a general class of stochastic processes characterized by some invariant properties of their multivariate probability distribution under scale change, which illustrate the empirically-observed scaling properties of rainfall time series.

The multifractal approach is based on the empirical detection of multifractal scale invariance of rainfall at a finite, but practically important, ranges of scales (Veneziano *et al.*

2006). In particular, in Section 2 we focus on multiplicative random cascade (MRC) models to construct discrete multifractal fields, which are extensively used in the literature (e.g., Gupta and Waymire 1993, Over and Gupta 1996, Menabde and Sivapalan 2000, Molnar and Burlando 2005, Gaume *et al.* 2007, Rupp *et al.* 2009, Serinaldi 2010, Licznar *et al.* 2011). The reason why MRC models have been so popular in the literature is that this method can parsimoniously generate complex intermittent and spiky patterns typical of rainfall time series, irrespective of whether the patterns are multifractal or not (Rupp *et al.* 2009).

The Hurst-Kolmogorov approach is based on the observation that: "Although in random events groups of high or low values do occur, their tendency to occur in natural events is greater" (Hurst 1951). This can be explained by multiple scales of changes within a stationary setting (Koutsoyiannis 2002). Moreover, this formalism distinguishes the different types of scaling behaviours: the scaling in state, which is related to the marginal distributional properties of the process (Koutsoyiannis 2005a), and the scaling in time, which is a property of the joint distribution function characterizing the time dependence structure of the process (Koutsoyiannis 2005b). Some multifractal analyses confuse the two. The proposed downscaling model following this approach (described in Section 3) is a simple method to generate time series based on logarithmic transformation of stepwise linear relationship from a Gaussian random process (Koutsoyiannis 2002).

2 MULTIPLICATIVE RANDOM CASCADE MODEL

2.1 Theoretical framework

Let $R_1^{(f)}$ be the average rainfall intensity over time scale f (equation (2)) at the time origin (j=1); $R_1^{(f)}$ is assumed to be a random variable with mean μ_0 and variance σ_0^2 of a stochastic process, which we wish to be stationary. $R_1^{(f)}$ (for convenience $R_{1,0}$) is then distributed over b sub-scale steps of equal size $\Delta s=f/b$ (i.e., $R_j^{(\Delta s)}$, j=1, 2, ..., b). This is accomplished by multiplying $R_{1,0}$ by b different weights (one for each sub-scale step) W which are independent and identically distributed (iid) random variables. Moreover, their distribution is assumed to be the same for all cascade levels with mean μ_W and variance σ_W^2 (Mandelbrot 1974).

After repeating this procedure k times (k cascade levels; k=0, 1, 2, …), the resulting discrete random process at the $\Delta s_k=b^{-k}f$ scale of aggregation can be expressed as:

$$R_j^{(\Delta s_k)} = R_{j,k} = R_{1,0}\prod_{i=0}^{k} W_{g(i,j),i} \quad (3)$$

where j=1, 2, …, b^k is the index of position in the series at level k; i is the index of the level of the cascade; $g(i, j)$ denotes a function which defines the position in the series at the level i, i.e. $g(i,j)=\left\lceil \frac{j}{b^{k-i}} \right\rceil$, which is a ceiling function (Gaume *et al.* 2007). For k=0 we have $W_{1,0}$=1.

For a *canonical* cascade (another common term to describe a downscaling model) the expected value of the mean process at the k-level is equal to the expected value of the process at the initial 0-level

$$\left\langle \frac{1}{b^k}\sum_{j=1}^{b^k} R_{j,k} \right\rangle = \langle R_0 \rangle \quad (4)$$

where < > denotes the expected value (i.e. average over the independent realizations of the stochastic process). The expected value of $R_{j,k}$ (equation (3)) is given by:

$$\langle R_{j,k} \rangle = \langle R_k \rangle = \left\langle R_{1,0}\prod_{i=0}^{k} W_{g(i,j),i} \right\rangle = \langle R_0 \rangle \prod_{i=0}^{k} \langle W_{g(i,j),i} \rangle = \langle R_0 \rangle \langle W \rangle^k = \mu_0 \mu_W^k \quad (5)$$

As a consequence of equations (4) and (5):

$$\frac{1}{b^k}\sum_{j=1}^{b^k}\langle R_k\rangle=\langle R_0\rangle,\quad \langle R_k\rangle=\mu_0,\quad \mu_0\mu_W^k=\mu_0,\quad \mu_W=1 \tag{6}$$

Thus, the weights W satisfy the condition μ_W=1.

For a *micro-canonical* cascade (i.e. a disaggregation model), the mean process at the k-level is equal to the process at the 0-level; this means that the following relationship (a consequence of (2)) holds for every pair of successive aggregation levels (k–1 and k) of the cascade:

$$\frac{1}{b}\sum_{i=b(j-1)+1}^{b\cdot j}R_{i,k}=R_{j,k-1} \tag{7}$$

where j=1,…, b^{k-1} with k>0. For example, if we choose b=2, then:

$$\frac{1}{2}\sum_{i=2j-1}^{2j}R_{i,k}=R_{j,k-1},\ R_{2j-1,k}+R_{2j,k}=2R_{j,k-1},\ W_{2j-1,k}=2-W_{2j,k} \tag{8}$$

Thus, the weights $W_{j,k}$ satisfy μ_W=1 and $W<b$ (e.g. in the case of equation (8), W<2). An important attribute of the micro-canonical model is that the distribution of W can be extracted from the data (Cârsteanu and Foufoula-Georgiou 1996), allowing a direct examination of the associations that the weights may have with other properties of rainfall.

2.2 Downscaling model (canonical cascade)

The summary statistics of the random process $R_{j,k}$ for a canonical cascade are given below. Specifically, we derive the variance, $\sigma_{j,k}^2$, the qth moment, $\langle R_{j,k}^q\rangle$, and the autocorrelation function for lag t, $\rho_{j,k}(t)$, of the random process at the k-level of the canonical cascade. The expected value, $\langle R_k\rangle$, has been already given in equation (5). The variance can be expressed as follows:

$$\sigma_{j,k}^2=\sigma_k^2=\langle R_{j,k}^2\rangle-\langle R_k\rangle^2=\left(\mu_0^2+\sigma_0^2\right)\left(1+\sigma_W^2\right)^k-\mu_0^2 \tag{9}$$

where the second moment is given by:

$$\begin{aligned}\langle R_{j,k}^2\rangle=\langle R_k^2\rangle=\left\langle R_{1,0}^2\prod_{i=0}^{k}W_{g(i,j),i}^2\right\rangle=\langle R_0^2\rangle\prod_{i=0}^{k}\langle W_{g(i,j),i}^2\rangle=\langle R_0^2\rangle\langle W^2\rangle^k=\\=\left(\mu_0^2+\sigma_0^2\right)\left(1+\sigma_W^2\right)^k\end{aligned} \tag{10}$$

Likewise, the qth moment is:

$$\langle R_{j,k}^q\rangle=\langle R_k^q\rangle=\langle R_{1,0}^q\rangle\langle W^q\rangle^k \tag{11}$$

Finally, the correlation coefficient for lag t is given by:

$$\rho_{j,k}(t)=\frac{\langle R_{j,k}R_{j+t,k}\rangle-\langle R_k\rangle^2}{\sigma_k^2}=\frac{\left(\mu_0^2+\sigma_0^2\right)\left(1+\sigma_W^2\right)^{h_{j,k}(t)}-\mu_0^2}{\left(\mu_0^2+\sigma_0^2\right)\left(1+\sigma_W^2\right)^k-\mu_0^2} \tag{12}$$

where the term $\langle R_{j,k}R_{j+t,k}\rangle$ can also be expressed as follows:

$$\langle R_{j,k}R_{j+t,k}\rangle=\langle R_0^2\rangle\langle W^2\rangle^{h_{j,k}(t)} \tag{13}$$

In equations (12) and (13), the exponent $h_{j,k}(t)$ (at the position j=1, …, b^k–t in the cascade at level k) is bounded in $\left[0,\ k-1-\lfloor\log_2 t\rfloor\right]$ if $0<t\le b^k-1$, where $\lfloor\ \rfloor$ denotes the floor function, while $h_{j,k}(t=0)=k$, for any j and k.

Assuming the cascade as a binary tree (b=2), the exponent $h_{j,k}(t)$ denotes the number of vertices of the tree (excluding the start vertex $R_{1,0}$) belonging to both simple paths leading to the vertices $R_{j,k}$ and $R_{j+t,k}$. The exponent $h_{j,k}(t)$ is computed as follows (see the explanatory sketch in Fig. 1):

$$h_{j,k}(t)=\begin{cases}\sum_{r=1}^{k}\Theta\left[2^{r-1}-j-t\right], & j\leq 2^{k-1}, t>0\\ h_{2^k-j-t+1,k}(t), & j>2^{k-1}, t>0\\ h_{2^k-j+1,k}(|t|), & t<0\end{cases} \tag{14}$$

where $\Theta[n]$ is the discrete form of the Heaviside step function, defined for a discrete variable (integer) n as:

$$\Theta[n]=\begin{cases}0, & n<0\\ 1, & n\geq 0\end{cases} \tag{15}$$

Thus, three important observations can be made. First, the exponent $h_{j,k}(t)$ is a function which satisfies a particular symmetry relation with respect to the position $j=2^{k-1}$ in the dyadic cascade at level k. Second, the autocorrelation function of a canonical MRC corresponds to a non-stationary process, because it depends on the position j in the cascade (i.e. the time position) for any level k. Third, we started assuming a stationary setting of the entire process at the largest scale, then we concluded with a downscaled process that we demonstrated to be non-stationary. Consequently, it can be argued that autocorrelograms produced by canonical MRC have a physically unrealistic attitude with respect to the rainfall process.

Although the derivation of the theoretical autocorrelation function presented in equations (12) and (14) is new, the problem of non-stationarity in processes generated by discrete random cascade models has been already discussed by Mandelbrot (1974, p. 356), who considered a canonical cascade with log-normal weights and a prescribed grid of eddies: "Because the eddies were prescribed, the random function [generated through the multiplicative scheme] is non-stationary and discontinuous: it varies between an eddy and its neighbors, by jumps that may be very large".

Moreover, this problem has been subsequently discussed by Over (1995), who highlighted the properties of non-stationarity (non-homogeneity) and anisotropy of the cross-moments of a discrete random cascade in a d-dimensional space, and by Veneziano and Langousis (2010, p. 137, Section 4.4.3.2). Hence, an important challenge is that of finding an alternative simple method to generate time series with spiky patterns typical of rainfall series and consistent with the observation at coarser scales, which is stationary. Indeed, as stated by Over (1995): "In applications, we may find that we want a random process model that is anisotropic and non-homogeneous, but in a way that is controllable using model parameters, not simply inherent to the model, and we would most likely want to use a homogeneous and isotropic model as a null hypothesis unless physical considerations determined otherwise" (p. 62, Section 3.4.1.1).

Thus, in Section 3 we propose a stationary downscaling model, based on the HKp, which is characterized by a cascade structure similar to that of MRC models.

2.2.1 Example: numerical simulation

In this section, numerical simulations of a canonical MRC are carried out. For simplicity and without loss of generality, we assume μ_0=1 and σ_0^2=0. Thus, the summary statistics given in the previous section (equations (5) and (9)-(12)) now become:

$$\langle R_k\rangle=1, \langle R_k^q\rangle=\langle W^q\rangle^k, \sigma_k^2=\left(1+\sigma_W^2\right)^k-1, \rho_{j,k}(t)=\frac{\left(1+\sigma_W^2\right)^{h_{j,k}(t)}-1}{\left(1+\sigma_W^2\right)^k-1} \tag{16}$$

This example refers to weights W log-normally distributed, defined as follows (see e.g. Over and Gupta 1996)

$$W = b^{\sigma_N Y - \frac{\sigma_N{}^2 \ln b}{2}} \qquad (17)$$

where Y is a normal $N(0,1)$ random variable; as a consequence, the variance of the weights is given by:

$$\sigma_W^2 = \exp\left(\sigma_N^2 (\ln b)^2\right) - 1 \qquad (18)$$

whereas $\sigma_N{}^2$ is a parameter defining the normal $N(-\sigma_N{}^2 \ln b^2/2,\ \sigma_N{}^2 \ln b^2)$ random variable $X = \ln W$.

Monte Carlo simulations (M=50 000) have been applied to explore the ensemble behaviour of the random process, assuming e.g. k=7 and σ_N=0.522, which gives $\sigma_k^2 = 1.5$ (from equation (16)).

Figures 2 and 3 show respectively the ensemble mean $\langle R_k \rangle$ and standard deviation σ_k of the random processes as a function of the position j along the cascade level k, j=1, 2, …, 2^k (we have $N=2^k=128$). Figure 4 shows how the ensemble autocorrelation function $\rho_{j,k}(t)$ strongly depends on the position j in the cascade at the level k.

In Fig. 5 (left), the autocorrelogram with starting point $j=N/2$ (midpoint of the cascade) is zoomed in the lag range [–5, 5] so as to illustrate that the lag 1 autocorrelation of the canonical MRC can be about 0.8 with the adjacent cell to the left and zero with the adjacent cell to the right. Moreover, if we move our simulation window just by two cells to the right, i.e. $j=N/2+2$ (see Fig. 5 right), then the lag 1 autocorrelation becomes about 0.8 and 0.6 with the adjacent cells to the left and to the right, respectively. These simple observations suffice to indicate how unrealistic and undesirable the stochastic structure of this model is.

2.3 Disaggregation model (micro-canonical cascade)

In the case of a micro-canonical cascade, the summary statistics of the random process $R_{j,k}$ can be expressed accounting for the equality constraint given in equation (7). The expected value $\langle R_k \rangle$, the variance $\sigma_k{}^2$ and the q-moments $\langle R_k^q \rangle$ remain the same as in the canonical case (equations (5) and (9)-(11)), while the autocorrelation function at lag t, $\rho_{j,k}(t)$, now becomes, for $t \neq 0$:

$$\rho_{j,k}(t) = \frac{\langle R_{j,k} R_{j+t,k} \rangle - \langle R_k \rangle^2}{\sigma_k^2} = \frac{\left(\mu_0^2 + \sigma_0^2\right)\left(1 + \sigma_W^2\right)^{h_{j,k}(t)}\left(1 - \sigma_W^2\right) - \mu_0^2}{\left(\mu_0^2 + \sigma_0^2\right)\left(1 + \sigma_W^2\right)^k - \mu_0^2} \qquad (19)$$

where the term $\langle R_{j,k} R_{j+t,k} \rangle$ can be also expressed as follows, if b=2 (equation (8)):

$$\langle R_{j,k} R_{j+t,k} \rangle = \langle R_0^2 \rangle \langle W^2 \rangle^{h_{j,k}(t)} \left(2 - \langle W^2 \rangle\right) \qquad (20)$$

Note that, when t=0, we have $h_{j,k}(0)=k$, for any j and k, and the term $(1-\sigma_W{}^2)$ in the numerator of equation (20) vanishes; consequently, we have $\rho_{j,k}(0)=1$. As in equations (12)-(13), the exponent $h_{j,k}(t)$ here also denotes the number of vertices of a binary tree (excluding the start vertex $R_{1,0}$) belonging to both simple paths leading to the vertices $R_{j,k}$ and $R_{j+t,k}$. The exponent $h_{j,k}(t)$ can still be computed by equation (14). Thus, the autocorrelation function of a micro-canonical MRC again corresponds to a non-stationary process, as in the canonical case.

2.4 Bounded random cascades

A special form of multiplicative random cascades is the bounded random cascade (Marshak *et al.* 1994). Bounded cascades allow the multiplicative weights W to depend on the cascade level k and converge to unity as the cascade proceeds; this implies that the simulated random process becomes smoother on smaller scales. In the literature, bounded random cascades have been frequently applied to the stochastic fine graining of rainfall observations into high resolution data both in the canonical and microcanonical form (e.g. Menabde *et al.* 1997, Menabde and Sivapalan 2000, Rupp *et al.* 2009, Licznar *et al.* 2011).

Bounded canonical cascades are constructed in the same way as the unbounded case, except that the weights W are iid only within a given cascade level, not among different levels as in the unbounded case (Menabde *et al.* 1997). Under these hypotheses and using the same notation as equation (13) above, the following holds:

$$\left\langle R_{j,k} R_{j+t,k} \right\rangle = \left\langle R_0^2 \right\rangle \prod_{i=0}^{h_{j,k}(t)} \left\langle W_{g(i,j),i}^2 \right\rangle \tag{21}$$

where, if $h_{j,k}(t)$=0 (no tree vertices in common) we have $W_{1,0}$=1 (see Section 2.1). Hence, the autocorrelation function of the time series generated by bounded canonical cascades still depends on the position j in the cascade level k.

3 HURST-KOLMOGOROV DOWNSCALING MODEL

In this section, we analyse a simple downscaling method to generate rainfall time series based on fractional Gaussian noise (e.g. Koutsoyiannis 2002 and references therein), also known as a Hurst-Kolmogorov process (HKp). The model disaggregates a fractional Gaussian noise by a dyadic additive cascade, which is then exponentially transformed to derive the actual rainfall time series that are consequently supposed to be log-normally distributed (e.g. Over 1995).

The HKp is a very simple and parsimonious stochastic process that represents the long-term persistence observed in many geophysical time series, such as rainfall time series. Furthermore, it has been demonstrated that such a stochastic representation is not solely data-driven, the Hurst-Kolmogorov behaviour emerges from extremal entropy production, which may provide a theoretical background for the HKp (Koutsoyiannis 2011).

The HK process can be defined as a stochastic stationary process which, for any integers i and j and any time scales f and l, has the property:

$$\left(\tilde{R}_j^{(f)} - \tilde{\mu}\right) \stackrel{d}{=} \left(\frac{f}{l}\right)^{H-1} \left(\tilde{R}_i^{(l)} - \tilde{\mu}\right) \tag{22}$$

where $\stackrel{d}{=}$ denotes equality in probability distributions, 0<H<1 is the Hurst coefficient. Typically, $\tilde{R}_j$ is Gaussian and $\tilde{\mu} = \left\langle \tilde{R} \right\rangle$ is its mean value. For the relevant process $\tilde{Z}_j^{(f)}$ the following holds (see equation (2)):

$$\mathrm{var}\left[\tilde{Z}_j^{(f)}\right] = f^2 \,\mathrm{var}\left[\tilde{R}_j^{(f)}\right] = f^{2H} \tilde{\sigma}^2 \tag{23}$$

where $\tilde{\sigma}^2 = \mathrm{var}[\tilde{R}]$. The autocorrelation function of either of $\tilde{R}_j^{(f)}$ and $\tilde{Z}_j^{(f)}$, for any aggregated time scale f, is a only function of the lag t and of the Hurst coefficient H (Koutsoyiannis 2002):

$$\tilde{\rho}^{(f)}(t) = \tilde{\rho}(t) = \frac{|t+1|^{2H}}{2} + \frac{|t-1|^{2H}}{2} - |t|^{2H} \tag{24}$$

3.1 Theoretical framework

Let $Z_1^{(f)}$ be the cumulative rainfall depth at the time origin (j=1) aggregated on the largest time scale f that is to be downscaled to a certain scale of interest. $Z_1^{(f)}$ is assumed to be a random variable with mean μ_0 and variance σ_0^2 of a stochastic process, which we wish to be stationary. We suppose the actual rainfall to be log-normally distributed.

Let us now introduce an auxiliary Gaussian random variable $\widetilde{Z}_1^{(f)} := \ln Z_1^{(f)}$ (for convenience $\widetilde{Z}_{1,0}$) of the aggregated HKp on the time scale f with mean $\widetilde{\mu}_0$ and variance $\widetilde{\sigma}_0^2$. It is well known that

$$\widetilde{\mu}_0 = \ln \mu_0 - \frac{1}{2} \ln\left(\frac{\sigma_0^2}{\mu_0^2} + 1 \right) \tag{25}$$

$$\widetilde{\sigma}_0^2 = \ln\left(\frac{\sigma_0^2}{\mu_0^2} + 1 \right) \tag{26}$$

$\widetilde{Z}_{1,0}$ is to be disaggregated by a dyadic (b=2) additive cascade. Then, $\widetilde{Z}_{1,0}$ is partitioned into two (b=2) Gaussian random variables on the time scale $\Delta s = f/2$; e.g. at the first cascade level (k=1) we have

$$\widetilde{Z}_{1,1} + \widetilde{Z}_{2,1} = \widetilde{Z}_{1,0} \tag{27}$$

Likewise, at the k-level, corresponding to the $\Delta s_k = 2^{-k} f$ scale of aggregation, we have

$$\widetilde{Z}_{2j-1,k} + \widetilde{Z}_{2j,k} = \widetilde{Z}_{j,k-1} \tag{28}$$

Thus, it suffices to generate $\widetilde{Z}_{2j-1,k}$ and then obtain $\widetilde{Z}_{2j,k}$ from equation (28) above.

This generic procedure resembles the well-known interpolation procedure, which is a point estimation. Thus, we can consider the following linear generation scheme (see the graphical example in Fig. 6):

$$\widetilde{Z}_{2j-1,k} = \boldsymbol{\theta}^T \boldsymbol{Y} + V \tag{29}$$

where $\boldsymbol{Y} = \left[\widetilde{Z}_{2j-3,k}, \widetilde{Z}_{2j-2,k}, \widetilde{Z}_{j,k-1}, \widetilde{Z}_{j+1,k-1} \right]^{\mathrm{T}}$, $\boldsymbol{\theta}$ is a vector of parameters, and V is a Gaussian white noise that represents an innovation term. Equation (29) allows the generated lower-level variable $\widetilde{Z}_{2j-1,k}$ to preserve autocorrelations with two earlier lower-level variables (level k) and one later higher-level variable (level k–1) (Koutsoyiannis 2002).

Koutsoyiannis (2001) demonstrated that the vector $\boldsymbol{\theta}$ which minimizes var[V] is of the form:

$$\boldsymbol{\theta} = \left\{ \mathrm{cov}[\boldsymbol{Y}, \boldsymbol{Y}] \right\}^{-1} \mathrm{cov}\left[\boldsymbol{Y}, \widetilde{Z}_{2j-1,k} \right] \tag{30}$$

Consequently, it can be shown that the least mean square prediction error of $\widetilde{Z}_{2j-1,k}$ from $\boldsymbol{Y}$ is the following:

$$\mathrm{var}[V] = \mathrm{var}\left[\widetilde{Z}_{2j-1,k} \right] - \mathrm{cov}\left[\widetilde{Z}_{2j-1,k}, \boldsymbol{Y} \right] \boldsymbol{\theta} \tag{31}$$

Hence, in each disaggregation step the two lower-level variables are generated by (equations (28)-(29)):

$$\begin{aligned} \widetilde{Z}_{2j-1,k} &= a_2 \widetilde{Z}_{2j-3,k} + a_1 \widetilde{Z}_{2j-2,k} + b_0 \widetilde{Z}_{j,k-1} + b_1 \widetilde{Z}_{j+1,k-1} + V \\ \widetilde{Z}_{2j,k} &= \widetilde{Z}_{j,k-1} - \widetilde{Z}_{2j-1,k} \end{aligned} \tag{32}$$

Parameters a_2, a_1, b_0 and b_1, and the variance of the innovation term V are estimated in terms of the correlation coefficients $\widetilde{\rho}(t)$, equation (24), which are independent of j and k, and of the variance of the HKp at the level k (Koutsoyiannis 2002), as given by equations (33) and (34):

$$\begin{bmatrix} a_2 \\ a_1 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 1 & \tilde{\rho}(1) & \tilde{\rho}(2)+\tilde{\rho}(3) & \tilde{\rho}(4)+\tilde{\rho}(5) \\ \tilde{\rho}(1) & 1 & \tilde{\rho}(1)+\tilde{\rho}(2) & \tilde{\rho}(3)+\tilde{\rho}(4) \\ \tilde{\rho}(2)+\tilde{\rho}(3) & \tilde{\rho}(1)+\tilde{\rho}(2) & 2[1+\tilde{\rho}(1)] & \tilde{\rho}(1)+2\tilde{\rho}(2)+\tilde{\rho}(3) \\ \tilde{\rho}(4)+\tilde{\rho}(5) & \tilde{\rho}(3)+\tilde{\rho}(4) & \tilde{\rho}(1)+2\tilde{\rho}(2)+\tilde{\rho}(3) & 2[1+\tilde{\rho}(1)] \end{bmatrix}^{-1} \begin{bmatrix} \tilde{\rho}(2) \\ \tilde{\rho}(1) \\ 1+\tilde{\rho}(1) \\ \tilde{\rho}(2)+\tilde{\rho}(3) \end{bmatrix} \tag{33}$$

and

$$\operatorname{var}[V] = \tilde{\sigma}_k^2 \left(1 - [\tilde{\rho}(2), \tilde{\rho}(1), 1+\tilde{\rho}(1), \tilde{\rho}(2)+\tilde{\rho}(3)]\,[a_2, a_1, b_0, b_1]^T\right) \tag{34}$$

Given equations (22) and (23), the mean and the variance of the HKp at the k-level of the cascade are:

$$\tilde{\mu}_k = \langle \tilde{Z}_{j.k} \rangle = \frac{\Delta s_k}{f} \tilde{\mu}_0 = \frac{\tilde{\mu}_0}{2^k} \tag{35}$$

$$\tilde{\sigma}_k^2 = \operatorname{var}[\tilde{Z}_{j.k}] = \left(\frac{\Delta s_k}{f}\right)^{2H} \tilde{\sigma}_0^2 = \frac{\tilde{\sigma}_0^2}{2^{2Hk}} \tag{36}$$

where $\Delta s_k = 2^{-k} f$.

The above stepwise disaggregation approach was first introduced by Koutsoyiannis (2002), who demonstrated that it effectively generates fractional Gaussian noise, but the rainfall process (especially at the resolution needed for hydrological applications) is not Gaussian. Indeed, we apply the following specific exponentiation to the HKp to make it log-normal but preserve its scaling properties (equations (35)-(36))

$$Z_{j,k} = \exp\left(\alpha(k)\tilde{Z}_{j,k} + \beta(k)\right) \tag{37}$$

In other words, we assume a unique HK process in the untransformed domain, and we change the characteristics of the transformed (exponentiated) domain using different characteristics for different disaggregation steps by means of the scale-dependent functions $\alpha(k)$ and $\beta(k)$:

$$\begin{aligned} \alpha(k) &= \frac{2^{Hk}}{\tilde{\sigma}_0} \sqrt{\ln\left(2^{2k(1-H)}\left(\exp\left(\tilde{\sigma}_0^2\right) - 1\right) + 1\right)} \\ \beta(k) &= -k \ln 2 - \tilde{\mu}_0 \left(\frac{\alpha(k)}{2^k} - 1\right) - \frac{\tilde{\sigma}_0^2}{2}\left(\frac{\alpha^2(k)}{2^{2Hk}} - 1\right) \end{aligned} \tag{38}$$

The mathematical derivation of these expressions of $\alpha(k)$ and $\beta(k)$ is given in the Appendix A.

The mean and variance of the log-normal variables $Z_{j,k}$ (actual downscaled rainfall) are

$$\mu_k = \langle Z_{j.k} \rangle = \frac{\mu_0}{2^k} \tag{39}$$

$$\sigma_k^2 = \operatorname{var}[Z_{j.k}] = \frac{\sigma_0^2}{2^{2Hk}} \tag{40}$$

while the autocorrelation function is given by

$$\rho_k(t) = \frac{\exp\left(\tilde{\sigma}_k^2 \tilde{\rho}(t)\right) - 1}{\exp\left(\tilde{\sigma}_k^2\right) - 1} \tag{41}$$

where $\tilde{\sigma}_k^2$ and $\tilde{\rho}(t)$ are given by equation (36) and (24) respectively.

The log-normality hypothesis and our specific exponential transformation (equations (37) and (38)) enable the analytical formulation of the main statistics of the actual rainfall process, given in equations (39)-(41), which are a key element for our theoretical analysis.

However, more elaborate normalizing transformations can be investigated (see, e.g. Papalexiou *et al*. 2011), but this is out of the scope of our paper.

The presented model is a disaggregation model only if the random variables are Gaussian; indeed, the equality constraint in equation (28) holds. However, under the hypothesis of log-normal rainfall, we have a downscaling model, where the lower-level rainfall time series generated are only statistically consistent with the given process $\boldsymbol{Z}$ at the coarser scale. The Hurst coefficient H is the only parameter of HK downscaling model.

3.2 Example: Numerical simulation

To further investigate the goodness of HK downscaling model, we explore its numerical simulations as we did for the MRC downscaling model in Section 2.2.1. To make the two model simulations comparable, we assume the same values of summary statistics as in the MRC case, i.e. k=7, $\mu_k = 1$ and $\sigma_k^2 = 1.5$. Furthermore, we assume H=0.7.

Figures 7 and 8 show, respectively, the behaviours of the ensemble mean $\langle R_k \rangle$ and standard deviation σ_k of the random processes as a function of the position j along the level k, j=1, 2, …, N (where $N = 2^k = 128$). Figure 9 shows how, unlike the MRC case, the ensemble autocorrelation function $\rho_{j,k}(t)$ is fully independent of the position in time j in the cascade at the level k. Thus, we verified that the process corresponding to the time series generated by the HK downscaling model is stationary.

3.3 Application to an historical observed event

In this section the HK downscaling model is fitted to an historical observed event, i.e. one of the Iowa events at the 10-second timescale (event 3); for further details on the observational data, the reader is referred to Georgakakos *et al.* (1994). The historical hyetograph is shown in Fig. 10 (upper panel). It can be seen that the dataset comprises a single storm without intermittence. Thus, intermittence, despite being an important characteristic of the rainfall process, can be left out of this analysis. We aim at providing further information on the applicability of the downscaling approach based on the HKp to reproduce the pattern of rainfall time series at the 10-second resolution.

We estimated the HK model parameter from the real data, which is H=0.92 (see also Koutsoyiannis *et al*. 2007). Figure 11 (upper panel), depicts the climacogram (i.e. a double logarithmic plot of the standard deviation of the aggregated process $\sigma(s)$ versus scale s) for both the real and the log-transformed datasets as a tool aiming at a multi-scale stochastic representation (see, e.g., equation (23)). It can be noticed that the two climacograms are approximately two parallel straight lines with high slopes ($H \cong 0.92$), which illustrates that the long-term persistence of the process is virtually invariant under a logarithmic transformation.

We performed 10 000 Monte Carlo experiments to downscale the aggregated rainfall event at the cascade level k=13. Figure 11 (lower panel) shows the 1st and 99th percentiles of climacograms for the HK downscaling model to highlight the scaling behaviour of the simulated time series, which is practically consistent with the scaling properties of the observed rainfall event. Figure 12 depicts a comparison between the observed autocorrelogram with that simulated by our model; in particular, we plot the 1st and 99th percentiles of autocorrelation function. It can be noticed that the observed behaviour is fitted quite satisfactorily by the model on average.

Finally, the historical hyetograph is compared (see Fig. 10) to two typical synthetic hyetographs, of equal length, generated by the MRC and the HK downscaling models (the MRC model parameters were estimated from the real data imposing both the mean and the variance of the lower-level variables). We can see that both models produce realistic traces without apparent visual differences in the general shapes from each other and from the real world hyetograph (note that the models provide copies with statistical resemblance but not precise reproductions of the historical event). Despite being visually similar, the study of the details of the statistical behaviour of the two models has revealed that there are important differences.

4 CONCLUSIONS AND DISCUSSION

The discrete MRC has been a widely-used approach of stochastic downscaling for rainfall time series. The usefulness of the discrete MRC relies on its simplicity and ability to generate time series characterized by both multifractal properties and complex intermittent and spiky patterns typical of rainfall time series.

By means of theoretical reasoning and Monte Carlo experiments, here we showed that the random process underlying the MRC model is not stationary, because its autocorrelation function is not a function of lag only, as it would be in stationary processes. Indeed, we provide a new theoretical formulation for the autocorrelation function of an unbounded canonical dyadic cascade, which is dependent on the lag, the position in time and the cascade level. As demonstrated, this undesirable violation of stationarity also extends to the micro-canonical and the bounded cascades. Consequently, MRC models cannot preserve joint statistical properties observed in real rainfall.

Mandelbrot (1974) made it clear that the structure of a discrete multiplicative cascade has problems. However, very many researchers miss this fact and treat these cascade models as if they were stationary (e.g. Menabde *et al.* 1997, Hingray and Ben Haha 2005, Gaume *et al.* 2007, Serinaldi 2010, Groppelli *et al.* 2011). Although fundamentally non-stationary, multiplicative random cascades were efficiently used to study the marginal and extreme distribution properties of stationary multifractal measures (see e.g. Veneziano *et al.* 2009 and references therein). Moreover, there exist other types of models intended to simulate multiscaling properties empirically observed in rainfall processes, which have been demonstrated to generate stationary processes, such as scale-continuous multifractal cascades (e.g., Lovejoy and Schertzer 2010a, b). However, this paper focuses on the analysis of discrete cascades, which are characterized by a very simple structure, easy to implement and, consequently, widely applied in the literature.

In this work, we propose and theoretically analyse an alternative downscaling approach based on the Hurst-Kolmogorov process (HKp), which is characterized by a simple cascade structure similar to that of MRC models, but it proves to be stationary. In its original formulation, this stepwise disaggregation approach effectively generates fractional Gaussian noise. However, the rainfall process (especially at the resolution needed for hydrological applications) is not Gaussian. Here we modified this approach to make it non-Gaussian by applying a logarithmic transformation to the time series generated, so as to make it a more realistic representation of the actual rainfall process and more comparable to the MRC models. However, the logarithmic normalizing transformation, which we chose for theoretical simplicity, is not the best choice to normalize the dataset (Papalexiou *et al*. 2011).

The HK downscaling model presented can be further developed in order to enable transformations different from the logarithmic, to account for intermittency, similar to MRC models (e.g., Over and Gupta 1996) and to simulate time series characterized by high values of the Hurst coefficient H. Specifically, for high H, the accuracy of the method in its current version could not be precise, but this could be remedied by expanding the number of lower- and higher-level variables that are considered in the generation procedure (equation (29)). Nonetheless, the current scheme is already good for any practical purpose.

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APPENDIX A

We apply a scale-dependent logarithmic transformation at the generic *k*-level of the cascade:

$$Z_{j,k} = \exp\left(\alpha(k)\widetilde{Z}_{j,k} + \beta(k)\right) \tag{A1}$$

where $\alpha(k)$ and $\beta(k)$ are scale dependent functions, which should be derived to preserve the scaling properties of the process $Z_{j,k}$ at different scales of aggregation.

The mean and the variance of the exponentiated process at the generic *k*-level of the cascade given in equation (A1) are:

$$\mu_k = \exp\left(\beta(k) + \alpha(k)\widetilde{\mu}_k + \alpha^2(k)\frac{\widetilde{\sigma}_k^2}{2}\right) \tag{A2}$$

$$\sigma_k^2 = \exp\left(2\beta(k) + 2\alpha(k)\widetilde{\mu}_k + \alpha^2(k)\widetilde{\sigma}_k^2\right)\left(\exp\left(\alpha^2(k)\widetilde{\sigma}_k^2\right) - 1\right) \tag{A3}$$

where $\widetilde{\mu}_k$ and $\widetilde{\sigma}_k^2$ are respectively the mean and the variance of the HKp at the cascade level *k*, given by equations (35) and (36). Substituting equations (35) and (36) in (A2) and (A3), respectively, we obtain

$$\mu_k = \exp\left(\beta(k) + \alpha(k)\frac{\widetilde{\mu}_0}{2^k} + \alpha^2(k)\frac{\widetilde{\sigma}_0^2}{2^{2Hk+1}}\right) \tag{A4}$$

$$\sigma_k^2 = \exp\left(2\beta(k) + 2\alpha(k)\frac{\widetilde{\mu}_0}{2^k} + \alpha^2(k)\frac{\widetilde{\sigma}_0^2}{2^{2Hk}}\right)\left(\exp\left(\alpha^2(k)\frac{\widetilde{\sigma}_0^2}{2^{2Hk}}\right) - 1\right) \tag{A5}$$

where $\widetilde{\mu}_0$ and $\widetilde{\sigma}_0^2$ are respectively the mean and the variance of the auxiliary normal variables $\widetilde{Z}_1^{(f)} = \widetilde{Z}_{1,0}$.

To derive the two functions $\alpha(k)$ and $\beta(k)$ we impose for the $Z_{j,k}$ process the same scaling laws of the relevant HKp ($\widetilde{Z}_{j,k}$)

$$\mu_k = \left\langle Z_{j.k} \right\rangle = \frac{\Delta s_k}{f}\mu_0 = \frac{\mu_0}{2^k} \tag{A6}$$

$$\sigma_k^2 = \mathrm{var}\left[Z_{j.k}\right] = \left(\frac{\Delta s_k}{f}\right)^{2H}\sigma_0^2 = \frac{\sigma_0^2}{2^{2Hk}} \tag{A7}$$

where μ_0 and σ_0 are respectively the mean and the standard deviation of the log-normal variables $Z_1^{(f)}$. Since we assume $Z_1^{(f)} = \exp\left(\widetilde{Z}_1^{(f)}\right)$, we have $\alpha(0)=1$ and $\beta(0)=0$ and, thus, equations (25) and (26) hold. Substituting equations (25) and (26) in (A6) and (A7) respectively, we obtain

$$\mu_k = \frac{1}{2^k}\exp\left(\widetilde{\mu}_0 + \frac{\widetilde{\sigma}_0^2}{2}\right) \tag{A8}$$

$$\sigma_k^2 = \frac{1}{2^{2Hk}}\exp\left(2\widetilde{\mu}_0 + \widetilde{\sigma}_0^2\right)\left(\exp\left(\widetilde{\sigma}_0^2\right) - 1\right) \tag{A9}$$

Equating the right-hand sides of equations (A4) and (A5) to (A8) and (A9), respectively, we obtain

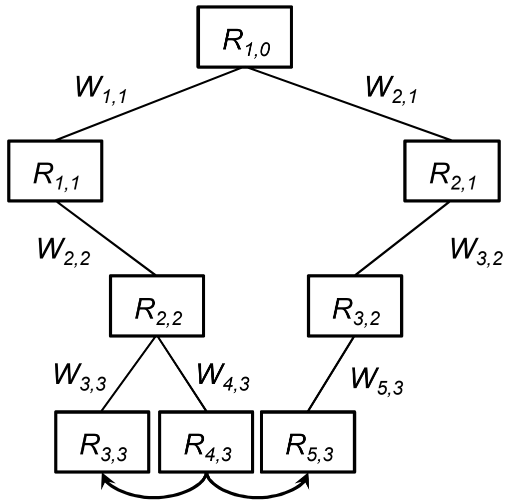
$$\exp\left(\beta(k) + \alpha(k)\frac{\widetilde{\mu}_0}{2^k} + \alpha^2(k)\frac{\widetilde{\sigma}_0^2}{2^{2Hk+1}}\right) = \frac{1}{2^k}\exp\left(\widetilde{\mu}_0 + \frac{\widetilde{\sigma}_0^2}{2}\right) \tag{A10}$$

$$\exp\left(2\beta(k) + 2\alpha(k)\frac{\widetilde{\mu}_0}{2^k} + \alpha^2(k)\frac{\widetilde{\sigma}_0^2}{2^{2Hk}}\right)\left(\exp\left(\alpha^2(k)\frac{\widetilde{\sigma}_0^2}{2^{2Hk}}\right) - 1\right) = \frac{1}{2^{2Hk}}\exp\left(2\widetilde{\mu}_0 + \widetilde{\sigma}_0^2\right)\left(\exp\left(\widetilde{\sigma}_0^2\right) - 1\right) \tag{A11}$$

Solving equations (A10) and (A11) we obtain equation (38).

FIGURES

$$\left\langle R_{j,k} R_{j+t,k} \right\rangle = \left\langle R_{1,0}^2 \right\rangle \left\langle W^2 \right\rangle^{h_{j,k}(t)}$$



$\boldsymbol{h_{4,3}(t=1)=0}$

$$\left\langle R_{4,3} R_{5,3} \right\rangle = \left\langle W_{4,3} W_{2,2} W_{1,1} W_{5,3} W_{3,2} W_{2,1} R_{1,0}^2 \right\rangle = \left\langle R_{1,0}^2 \right\rangle$$

$\boldsymbol{h_{4,3}(t=-1)=2}$

$$\left\langle R_{4,3} R_{3,3} \right\rangle = \left\langle W_{4,3} W_{2,2}^2 W_{1,1}^2 W_{3,3} R_{1,0}^2 \right\rangle = \left\langle R_{1,0}^2 \right\rangle \left\langle W^2 \right\rangle^2$$

$$R_{3,3} = W_{3,3} W_{2,2} W_{1,1} R_{1,0} \qquad R_{4,3} = W_{4,3} W_{2,2} W_{1,1} R_{1,0} \qquad R_{5,3} = W_{5,3} W_{3,2} W_{2,1} R_{1,0}$$

Fig. 1: Example of computation of the exponent $h_{j,k}(t)$ for a canonical MRC. In the computation we use equation (13) and the arrows indicate the links to those variables considered.

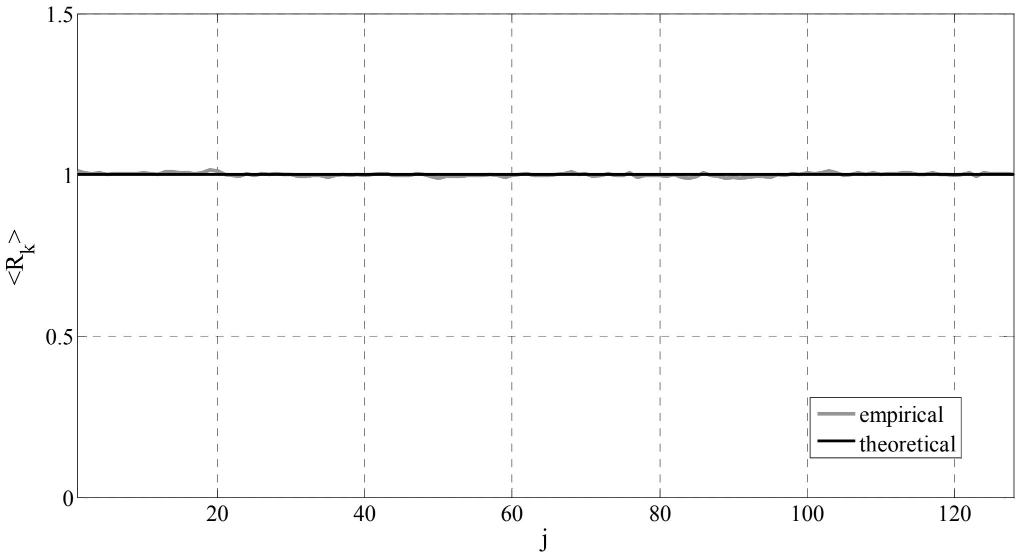


Fig. 2: Ensemble mean of the example MRC process as a function of the position j along the cascade level k = 7.

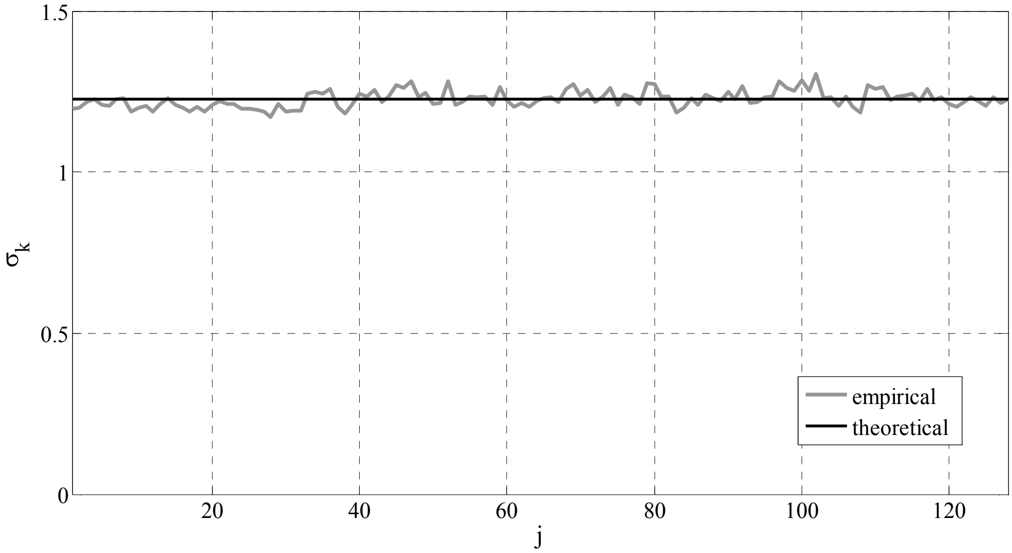


Fig. 3: Ensemble standard deviation of the example MRC process as a function of the position *j* along the cascade level $k = 7$.

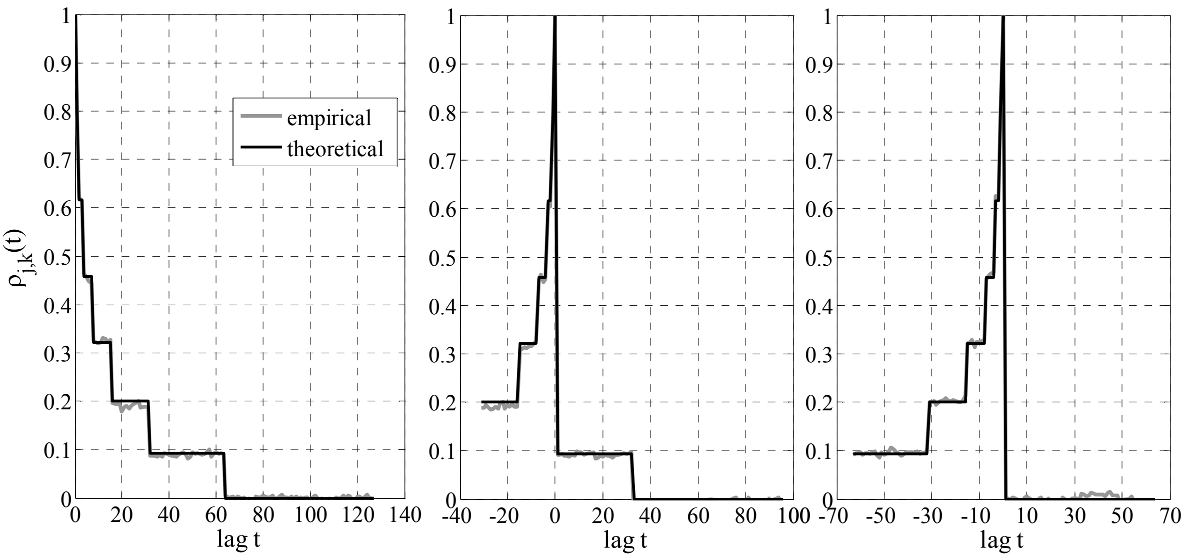


Fig. 4: Ensemble autocorrelation function of the example MRC process at the cascade level $k = 7$ with starting point *j* (for $j = 1$, $N/4$ and $N/2$, respectively, from left to right) in the considered cascade level with $N = 2^7 = 128$ elements.

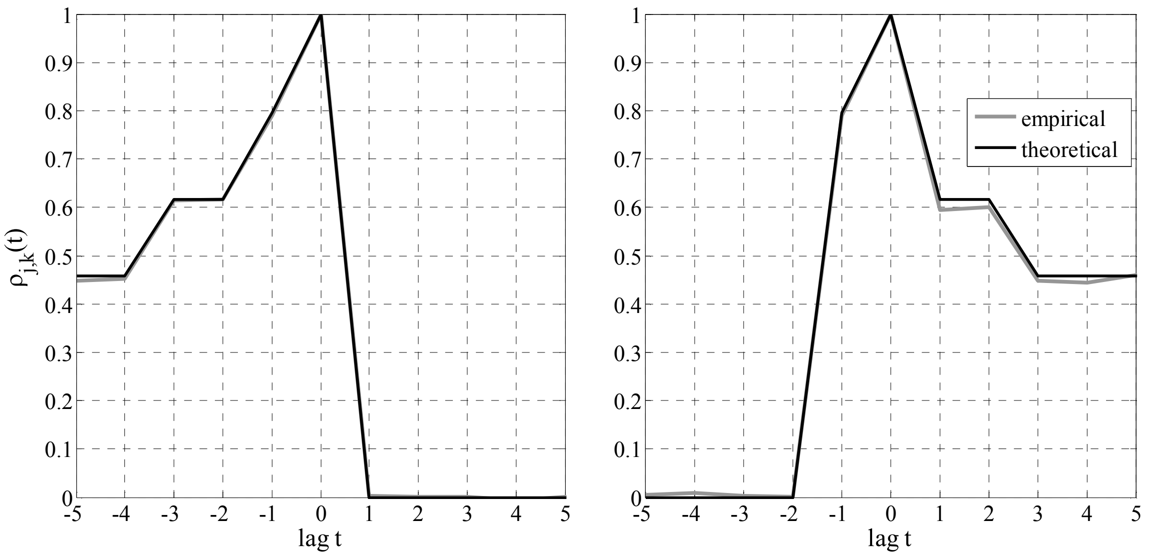


Fig. 5: Ensemble autocorrelation function of the example MRC process at the cascade level k = 7 with starting point $j = N/2$ (left) and $j = N/2 + 2$ (right) zoomed at the lag range [–5, 5].

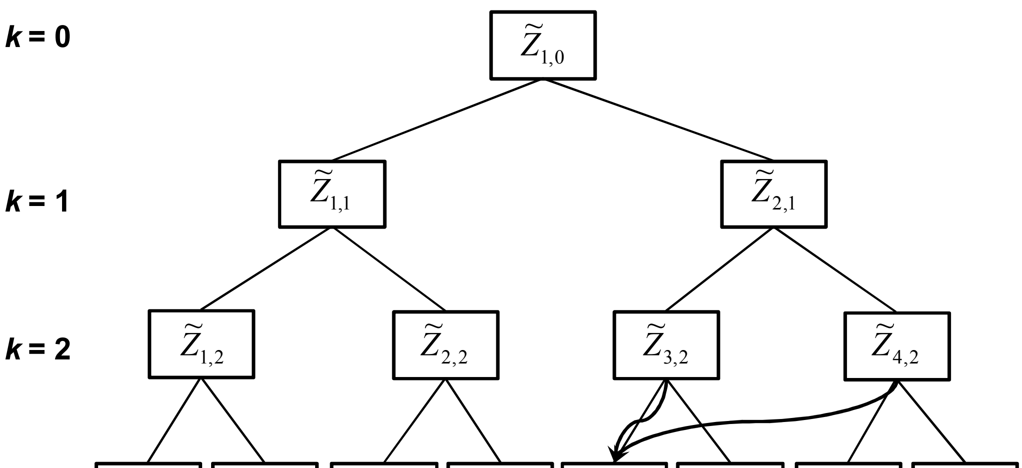


Fig. 6: Example of the dyadic additive cascade for four disaggregation levels (k = 0, 1, 2, 3), where arrows indicate the links to those variables considered in the current generation step (adapted from Koutsoyiannis 2002).

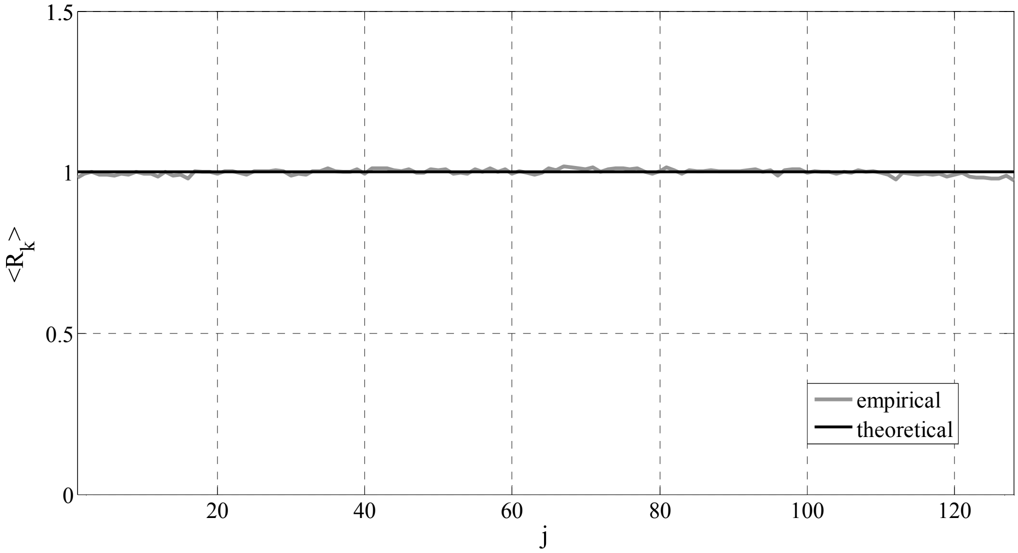


Fig. 7: Ensemble mean of the example HK process as a function of the position *j* along the cascade level $k = 7$.

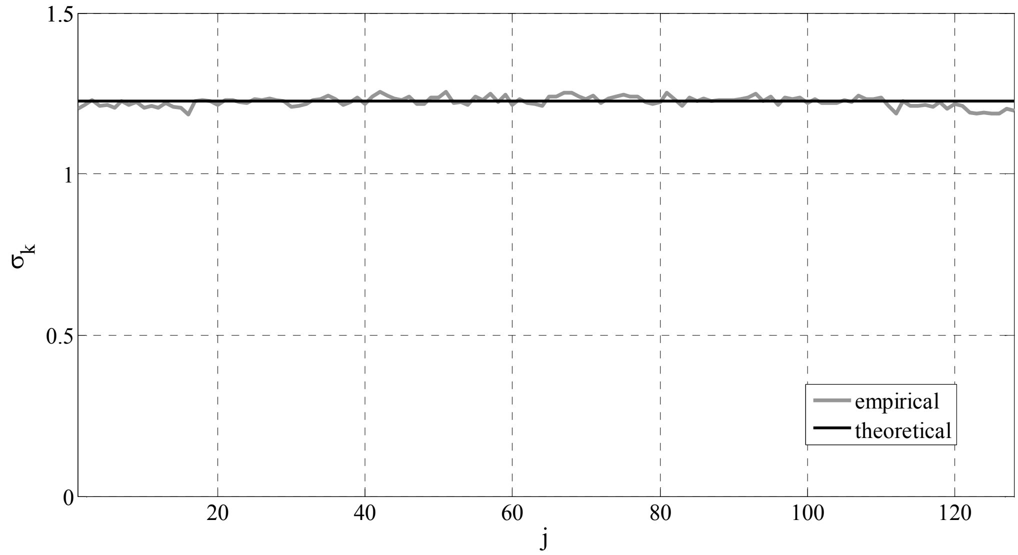


Fig. 8: Ensemble standard deviation of the example HK process as a function of the position *j* along the cascade level $k = 7$.

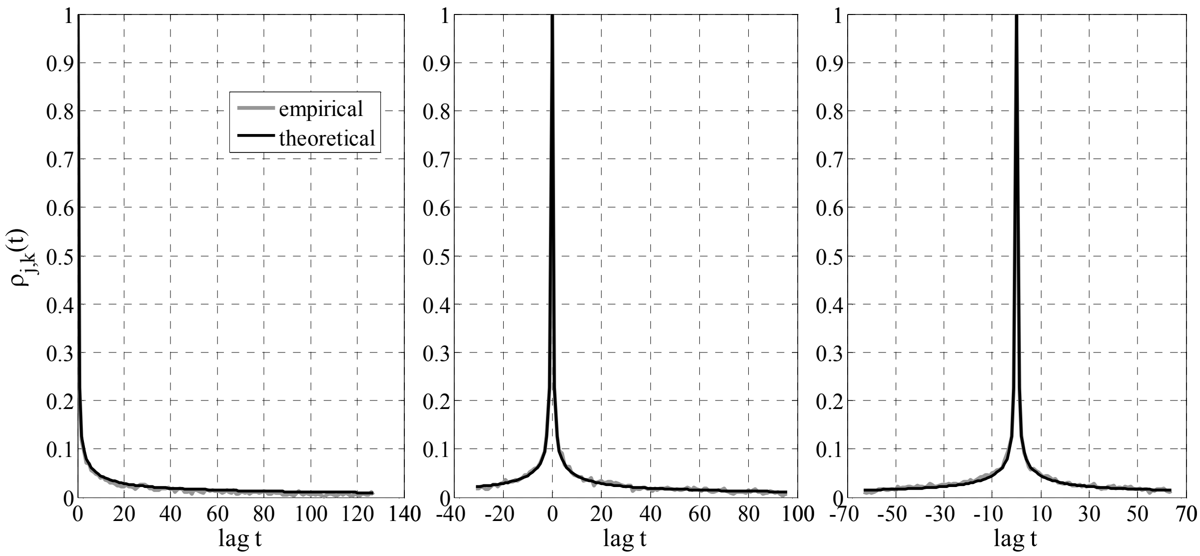


Fig. 9: Ensemble autocorrelation function of the example HK process at the cascade level $k = 7$ with starting point j (for $j = 1$, $N/4$ and $N/2$, respectively, from left to right) in the considered cascade level with $N = 2^7 = 128$ elements.

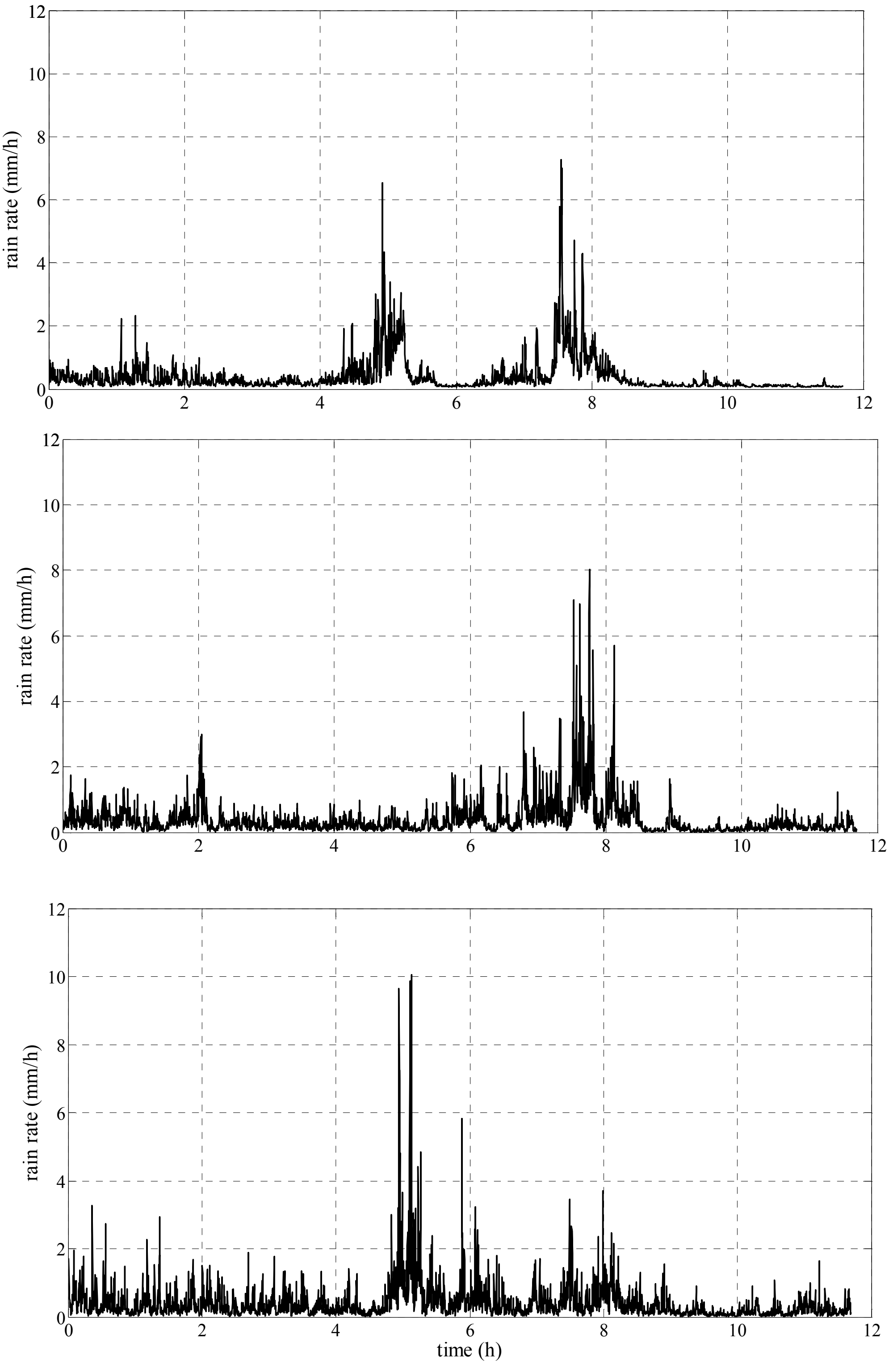


Fig. 10: Hyetograph of the historical rainfall event (no. 3) measured in Iowa on the November 30th 1990 (upper panel; Georgakakos *et al.*, 1994) along with two synthetic time series of equal length generated by the MRC and HK models (middle and lower panels, respectively).

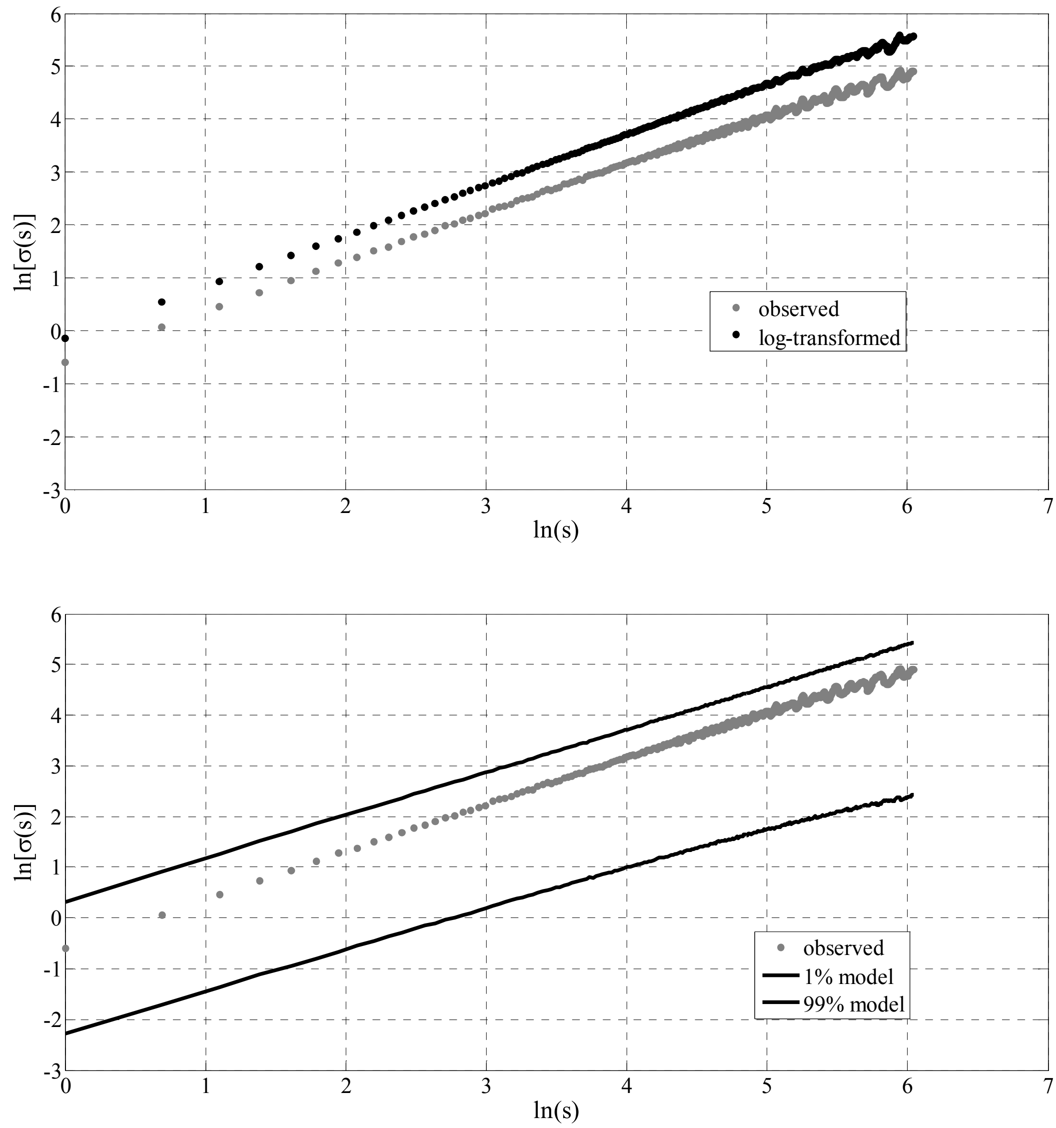


Fig. 11: Double logarithmic plot of the standard deviation of the aggregated process $\sigma(s)$ vs. scale s (climacogram) for both the real and the log-transformed data of the Iowa rainfall event (upper panel); climacograms of the 1st and the 99th percentiles for the HK downscaling model (10 000 Monte Carlo experiments) and for the observed rainfall event (lower panel).

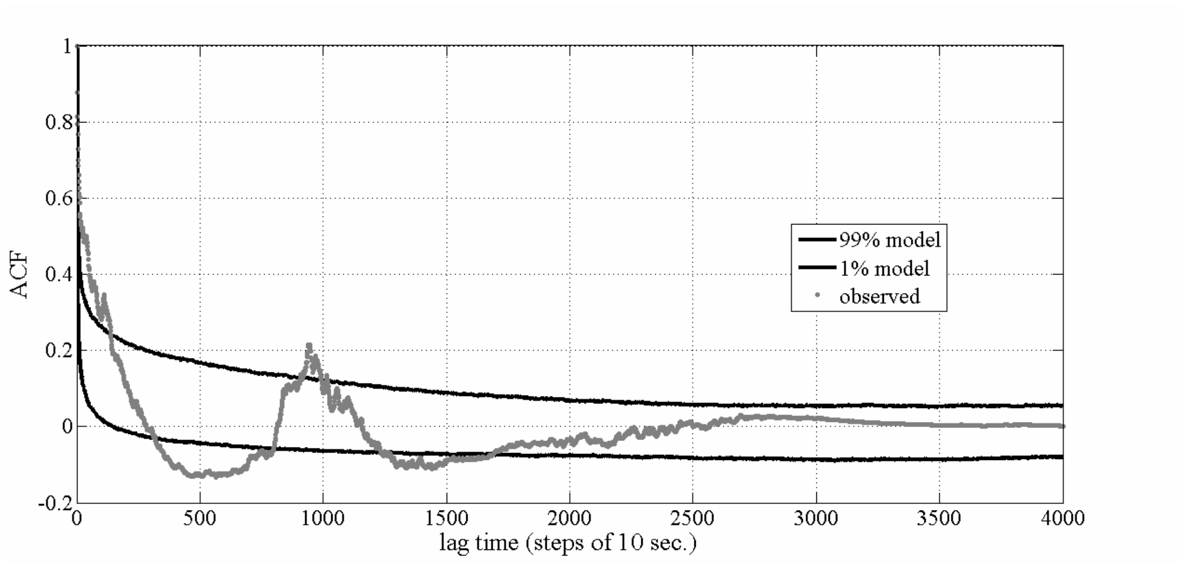


Fig. 12: Empirical autocorrelation function (ACF) of the Iowa rainfall event examined and 1st and 99th percentiles of ACF for the HK downscaling model.