

## ***Interactive comment on “Just two moments! A cautionary note against use of high-order moments in multifractal models in hydrology” by F. Lombardo et al.***

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### **Further (monofractal) Limitations of Climactograms**

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In the comment “Multifractality Requires At Least Three Parameters!” (hereafter SLT) it was pointed out that due to multifractality, exponents other than second order were required to characterize the statistics of scaling processes. But beyond this, using the authors’ “climactogram”, there is serious problem with the determination of the even the second order moments, even in quasi-Gaussian processes where there is no multifractality (this includes fractional Brownian motion, here rebaptized as Hurst-Kolmogorov processes)! We would therefore like to take this opportunity to make a short comment about additional serious limitations of the climactogram and point out a straightforward alternative, Haar fluctuations and structure functions.

As the authors point out, for scaling processes, the standard deviation of the process at a degraded resolution (previously called the Aggregated Standard Deviation (ASD, [Koutsoyiannis and Montanari, 2007])), now called the “climactogram” is only related to the autocorrelation and the spectrum for a narrow range of scaling exponents  $0 < H_{clim} < 1$  (the authors’ exponent “H” that we denote “ $H_{clim}$ ” is the same as H in SLT). For scaling, Gaussian processes (e.g. in 1-D, a Gaussian white noise filtered by  $\omega^{-\beta/2}$  where  $\omega$  is the frequency), when for  $-1 < \beta < 1$  this corresponds to spectral exponent  $\beta = 2H_{clim} - 1$ . When  $\beta$  is outside this range, then the relationship between  $\beta$  and  $H_{clim}$  breaks down. The reason is simple: the fluctuations at scale  $\Delta t$  are no longer dominated by frequencies  $1/\Delta t$  so that spectral and real space exponents are no longer linked. In particular, if  $\beta > 1$ , then the autocorrelation is dominated by the lowest

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frequencies present in the series. Although for finite data, one obtains a finite result, it is spurious in the sense that it depends on the details of the series, in particular, its length. Any useful scale information is lost. The problem with the climactogram - independently of the fact that it is inadequate for multifractal processes - is therefore that if it is applied to series with  $\beta > 1$  then the scaling exponents will be spurious. However this is by no means an exceptional situation, on the contrary,  $\beta > 1$ ,  $H > 0$ , is quite generally the case for atmospheric fields at weather scales (in space up to the size of the planet, in time up to about 10 days), as well as at climate scales (longer than 10-30 years). In fact, the only atmospheric regime where the climactogram will give useful estimates is in the intermediate macroweather regime where atmospheric fields generally have  $-1 < H < 0$  so that  $0 < H_{clim} < 1$ ; indeed this is the regime where it has been most successfully applied. In addition, for climactogram applications, it is fortuitous that empirically the macroweather regime is also characterized by weak multifractality [Lovejoy and Schertzer, 2013].

To better understand the limitations of the climactogram, it is useful to put it in the framework of wavelet analysis. In this case, the climactogram corresponds to the second order moment ("structure function") of the tendency fluctuation  $\Delta X$ :

$$(\Delta X(t, \Delta t))_{tend} = \frac{1}{\Delta t} \int_t^{t+\Delta t} X'(t') dt'; \quad X' = X - E(X) \quad (1)$$

where  $(\Delta X(t, \Delta t))_{tend}$  is the tendency function at time  $t$ , lag (scale)  $\Delta t$ ,  $E$  indicates "expectation". In a scaling regime we have:

$$E((\Delta X(t, \Delta t))_{tend}^q) \propto \Delta t^{\xi(q)}; \quad \xi(q) = qH - K_n(q) \quad (2)$$

where  $\xi(q)$  is the generalized structure function exponent and we have used  $K_n(q)$ , the nonlinear part of the exponent  $K(q)$  of a pure conservative multiplicative cascade which satisfies  $K_n(q) = 1$ . For  $q = 2$ , the above is the climactogram defined in eq.4 of the discussion paper. As pointed out in [Lovejoy and Schertzer, 2012; 2013],  $H_{clim} = 1 +$

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$\xi(2)/2$  so that  $H_{clim} = H + 1 - K_n(2)/2$  and  $K_n(q)$  are due to multifractal intermittency (see STL). However, to simplify the discussion, here we will ignore this and focus on the linear scaling part, the exponent  $H$ .

As mentioned, for some applications - such as macroweather - the climactogram may be adequate. However, for studying other regimes, we need a tool that is useful over a wider range of exponents. Traditionally, one uses fluctuations defined as differences:

$$(\Delta X(t, \Delta t))_{diff} = X(t + \Delta t) - X(t) \quad (3)$$

(the "poor man's wavelet"), the moments  $E((\Delta X(t, \Delta t))_{diff}^q)$  being the usual structure functions ( $q = 2$ ; "generalized structure functions"  $q \neq 2$ ). However this is only useful for  $0 < H < 1$  ( $1 < \beta < 3$ ; recall that "useful" means that the fluctuations at scale  $\Delta t$  are dominated by frequencies  $1/\Delta t$  so that the spectral and real space exponents are linked). In [Lovejoy and Schertzer, 2012] it was pointed out that a single tool - the Haar fluctuation - conveniently covers the whole range  $-1 < H < 1$  yet remains simple to calculate and interpret:

$$(\Delta X(t, \Delta t))_{Haar} = \frac{2}{\Delta t} \int_t^{t+\Delta t/2} X(t') dt' - \frac{2}{\Delta t} \int_{t+\Delta t/2}^{t+\Delta t} X(t') dt' \quad (4)$$

i.e  $(\Delta X(t, \Delta t))_{Haar}$  is simply the difference between the means of the first and second halves of the interval  $\Delta t$ . It is easy to show that in scaling regimes that:

$$E((\Delta X(t, \Delta t))_{tend}^q) \propto E((\Delta X(t, \Delta t))_{Haar}^q); \quad -1 < H < 0 \quad (5)$$

$$E((\Delta X(t, \Delta t))_{diff}^q) \propto E((\Delta X(t, \Delta t))_{Haar}^q); \quad 0 < H < 1 \quad (6)$$

Therefore, the climactogram could be considered as an (unnecessary) restriction of the Haar fluctuations to monofractality ( $q = 2$ ) and to processes with mean fluctuations that decrease with scale ( $-1 < H < 0$ ). In addition, if needed, it is easy to extend the Haar fluctuations so as to cover arbitrarily wide ranges of exponents (although if this is done, the simplicity of the interpretation is lost).

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## 1 References

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