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Research article

Stochastic assessment of temperature–CO_2 causal relationship in climate from the Phanerozoic through modern times

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Appendix

Appendix A: Adaptation of IRF coordinates in the case of using a time step multiplier

Given the IRF coordinates g_i, we can readily calculate the coordinates of the adapted one, g'_j, either by a rounding operation or a linear interpolation. In the first of the two alternatives we have:

$$g'_j := g_{[j/m]} \tag{A1}$$

with $[a]$ denoting the standard rounding function of the real number a. In other words, in this option we round the number j/m and assign g'_j as the g_i for $i = [j/m]$.

In the second option we perform linear interpolation between two g_i values above and below j/m, i.e.,

$$g'_j := \{j/m\} g_{\lceil j/m \rceil} + (1 - \{j/m\}) g_{\lfloor j/m \rfloor} \tag{A2}$$

with $\lceil a \rceil, \lfloor a \rfloor, \{a\}$ denoting, respectively, the ceiling, the floor and the fractional part of the real number a.

Appendix B: Theoretical investigation on characteristic time lags

To illustrate the effect of the observation time window on the estimated characteristic lags, we assume that the true IRF is of Pareto type,

$$g(h) := (1+\xi h)^{-1-\frac{1}{\xi}} \tag{A3}$$

with tail index $\xi = 2$. We assume that, while in reality the lag h spans from 0 to ∞, we only have an observational window of length W and we truncate Equation (A3) to the interval [0, W]. By integration it is found that the truncated mean will be

$$\mu_h^W = \frac{(1+W)(1+\xi W)^{-1/\xi} - 1}{\xi - 1} \tag{A4}$$

while the truncated median can be found from

$$\int_0^{h_{1/2}^W} g(x)\mathrm{d}x = \frac{1}{2}\int_0^W g(x)\mathrm{d}x \tag{A5}$$

which, when solved, yields

$$h_{1/2}^W = \frac{2^\xi\left(1+(1+\xi W)^{-1/\xi}\right)^{-\xi} - 1}{\xi} \tag{A6}$$

These are shown graphically in Figure A1.

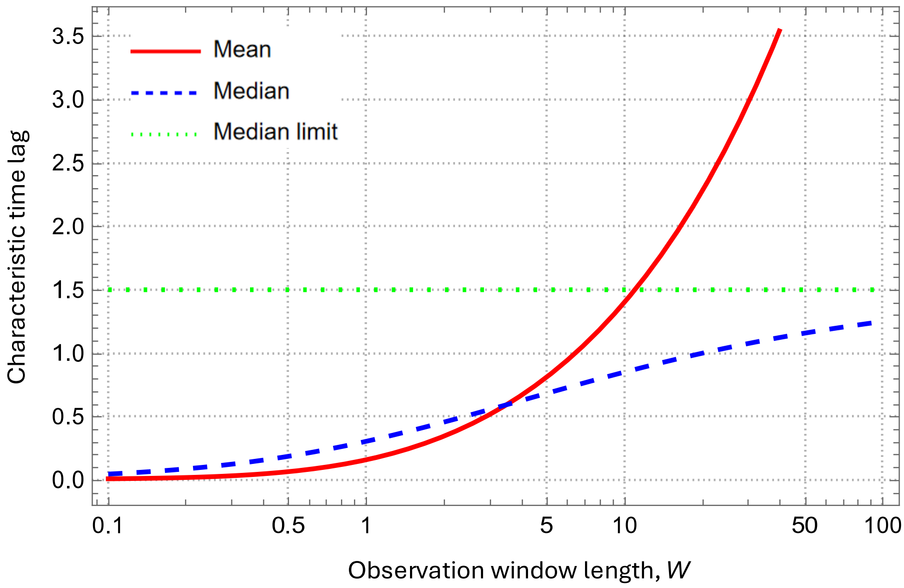

Figure A1. Characteristic time lags for a truncated Pareto IRF as functions of the window length.

To find the true characteristic time lags we take the limits as $W \to \infty$ and obtain:

$$\lim_{W\to\infty} \mu_h^W = \infty, \qquad \lim_{W\to\infty} h_{1/2}^W = \frac{2^\xi - 1}{\xi} \tag{A7}$$

and for $\xi = 2$:

$$\lim_{W\to\infty} \mu_h^W = \infty, \qquad \lim_{W\to\infty} h_{1/2}^W = \frac{3}{2} \tag{A8}$$

Apparently, the infinite mean cannot be recovered by any observation window length W, while even the median converges slowly to its theoretical value of 1.5. These observations justify why the characteristic time lags estimated by our stochastic method can increase as the observation window length increases.

As already mentioned, the lag that maximizes the cross-correlation does not depend on the IRF if a time series is given. However, in a theoretical level, if both the IRF, $g(h)$, and the autocorrelation function, $c_{xx}(h)$, are identified, then the cross-correlation function is [1]:

$$c_{yx}(h) = \int_0^\infty g(a) c_{xx}(h-a) \mathrm{d}a = \int_{-\infty}^h c_{xx}(b) g(h-b)\, \mathrm{d}b \tag{A9}$$

and by maximizing it we can determine the lag in question. By virtue of the symmetry of $c_{xx}(h)$ we can write

$$c_{yx}(h) = \int_0^\infty c_{xx}(y) g(h+y)\, dy + \int_0^h c_{xx}(y) g(h-y)\, dy \tag{A10}$$

As an illustration, we may again use the example of Equation (A3) with a power-type autocorrelation function,

$$c_{xx}(h) = \left(1 + \frac{h}{\lambda}\right)^{2H-2} \tag{A11}$$

where H is the Hurst parameter and λ is a scale parameter. As an additional case, the exponential (Markov) autocorrelation is also examined:

$$c_{xx}(h) = e^{-\frac{h}{\lambda}} \tag{A12}$$

For particular values of H, analytical solution of Equation (A10), are possible. For example, for $H = 3/4$ (and $\xi = 2$) we get:

$$c_{yx}(h) = \frac{\sqrt{2}(1+2h)\sqrt{\lambda} + 2\sqrt{2}\lambda^{3/2} + 2(1+2h)\sqrt{\lambda(h+\lambda)} - 4\lambda(\sqrt{1+2h} + \sqrt{\lambda(h+\lambda)})}{(1+2h)^2 - 4\lambda^2} \tag{A13}$$

However, numerical integration and maximization to find the maximizing lag h_c is always possible. Figure A2 exemplifies several cases, and also shows the true median, which, as found above is 1.5. The figure shows that h_c can substantially differ from $h_{1/2}$ and, hence, the differences found in real-world applications should not need to trouble us.

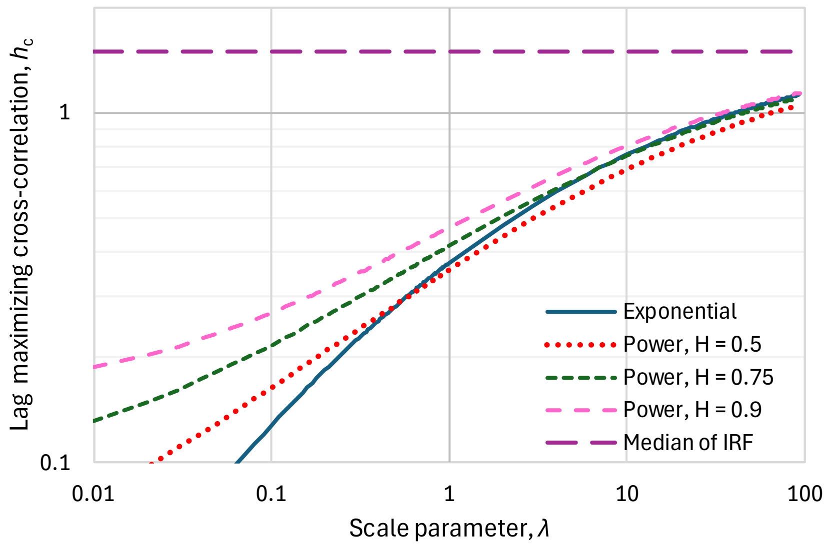

Figure A2. An example of the variation of the time lag maximizing the cross-correlation coefficient as a function of the scale parameter of the autocorrelation function.