1. Abstract
We investigate three methods for simultaneous estimation of the Hurst parameter ($H$) and the standard deviation ($\sigma$) for a Hurst-Kolmogorov stochastic process, namely the maximum likelihood method and two methods based on the variation of the standard deviation or of the variance with time scale. We show that the simultaneous estimation of the two parameters is important, albeit not given appropriate attention in the literature, because of the interdependence of the two parameter estimators. In addition, we test the performance of the three methods for a range of sample sizes and $H$ values, through a simulation study and we compare it with other known results for other estimators of the literature.

2. Definition of Hurst-Kolmogorov process
- Let $X_t$ be a Gaussian stationary stochastic process with $\mu = 0$, $\sigma = 1$, $\alpha = 0.5$, and $\beta = 0.5$. We form the vector of identically distributed variables $X_i = (X_1, \ldots, X_N)$.
- The mean process is $\mu = E[X_t]$ and the variance of the process is $\sigma^2 = Var[X_t]$.
- Let $k$ be an integer that represents a time scale larger than 1, the original time scale of the process $X_t$. The mean aggregated stochastic process on that time scale is $X^k = (1/k)^{1/2} \sum_{i=1}^{k} X_i$.
- The following equation defines the Hurst-Kolmogorov process (HKp):
  \[ (2-k^2) \rho_1 (x^k - x^0) = 0 \text{ for } k = 1, 2, \ldots \]
- The autocorrelation function of the mean aggregated stochastic process for any aggregated timescale $k$ is independent of $k$ and given by the following equation:
  \[ \rho_1 = \rho_0 = 1 - \frac{1}{2} k^2 (2 - k^2)^{3/2} \]

3. The maximum likelihood (ML) estimator
The likelihood function to be maximized is:
\[ L(\alpha, \beta) = \prod_{i=1}^{N} p_{\alpha, \beta}(X_i) \]
This function is maximized when $\alpha = 1$ and $\beta = 0.5$ because $\beta$ is a probability density function and $\alpha = 0$.

4. The Least Squares based on Standard Deviation (LSSD) estimator

5. The Least Squares based on Variance (LSV) estimator

6. Comparison of all three estimators

7. Comparison between ML, LSSD, LSV and alternative methods

8. Influence of $H$ to the estimators’ performance

9. Interdependence between the $H$ and $\sigma$ estimators

10. Conclusions
Three estimators (ML, LSSD, LSV) relying on the structure of the HK stochastic process are used to estimate its parameters. These estimators have the advantage to be more accurate when compared to the usual estimators of the literature. The finite sample properties of these estimators are explored.

11. Appendix – Theoretical results (1)
Derivation of the ML estimator

12. Appendix – Theoretical results (2)
Bracketing of $H$ for the LSV estimator
Suppose that $\text{E}(\hat{H}) = 0$ and $\text{E}(\hat{\sigma}^2) < 0$ to prove that an estimated $\hat{\sigma}^2 < 0$ always. New for any $H \in (0, 1)$ we can always find a $\alpha < 1$, such that $p_{\alpha}(x^k, X_0) = \hat{\sigma}^2 < 0$ for every $k$. For these values of $H$, and $\beta$, $p_{\alpha}(x^k, X_0) = \hat{\sigma}^2 < 0$ for every $k$. This proves that $\text{E}(\hat{H}) / \text{E}(\hat{\sigma}^2) > (0, \infty)$. So $\hat{\sigma}^2$ remains its minimum for $H < 1$. This property is important since other estimators permit estimated values of $\hat{H}$ higher than 1, for which a HK process cannot be defined.

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