CHAPTER 4

SCALING AND FRACTALS IN HYDROLOGY

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Virtually, all areas of hydrology have been deeply influenced by the concepts of fractality and scale invariance. The roots of scale invariance in hydrology can be traced to the pioneering work of Horton, Shreve, Hack and Hurst on the topology and metric properties of river networks and on river flow. This early work uncovered symmetries and laws that only later were recognized as manifestations of scale invariance. Le Cam, who in the early 1960s pioneered the development of multi-scale pulse models of rainfall, provided renewed impetus to the use of scale-based models. Fractal approaches in hydrology have become more rigorous and widespread since Mandelbrot systematized fractal geometry and multifractal processes were discovered. This chapter reviews the main concepts of fractality and scale invariance, the construction of scale-invariant processes, their properties, and the inference of scale invariance from data. We highlight the recent developments in four areas of hydrology: rainfall, fluvial erosion topography, river floods, and flow through porous media.

1. Introduction

This chapter deals with fractal methods and their importance in hydrology. By fractal methods, we mean models, analysis and inference procedures that emphasize scale invariance, which is the property that an object reproduces itself under some scale-change transformation.

Like other forms of invariance (invariance under translation called stationarity; invariance under translation, rotation, and reflection called isotropy), invariance under a change of scale is a fundamental property that entities like sets, functions, or measures may have. When they do, this form of invariance often sheds light on the genesis of the phenomenon, reduces the complexity of models and their inference, and allows one to devise special methods to upscale/downscale, characterize extremes, make predictions, etc. Our interest in scale invariance stems from this deep understanding and richness of applications.

In the physical world, scale invariance rarely manifests itself as a deterministic property, whereby an object is made of exact scaled replicas of itself. Rather, nature displays variability that is generally best described through stochastic models. For this reason, here we deal exclusively with random objects that are scale-invariant in the sense that their probability laws (their ensemble statistical properties) do not change under certain scale-change transformations. Specific realizations of the object are not expected to display deterministic scale invariance.

Even in the limited context of hydrology, the use of scale invariance has grown very significantly over the years, making it difficult to provide a comprehensive coverage in book-chapter form. Entire books (e.g. Rodriguez-Iturbe and Rinaldo[1]) exist on just specific application areas. This rapid growth, together with the fact that the authors are more intimately familiar with certain areas, necessarily results in a personal topic selection and presentation style. The material itself is often drawn from our previous publications.

This review emphasizes stochastic (random process) approaches to scale invariance. For an alternative approach through chaos theory, see Barnsley[2] and for applications of scale-invariant chaotic models in hydrology see, for example, Puente and Obregón[3] and Puente and Sivakumar[4].

Sections 2, 3, and 4 contain introductory material: Section 2 deals with fractal and scale-invariant sets, their fractal dimension, and generation. Section 3 presents various scale-invariance conditions for random processes (ordinary functions and generalized functions or measures), drawing a distinction between self-similarity and multifractality. Section 4 is devoted to the characterization and generation of scale-invariant processes, including their relationship with stationary processes and their generation as renormalization limits of other processes or as weighted sums of processes at different scales. Deviations from exact scaling are briefly covered at the end of Section 4. Section 5 deals with basic properties of scale-invariant processes. Some properties, such as moment scaling, hold irrespective of whether the process is stationary or not, whereas other properties (e.g. marginal distributions, extremes) are more specific to stationary random measures; this explains the emphasis on stationary measures in Section 5. Section 6 covers two frequently needed operations with stationary multifractal measures: forecasting using observations from the past and downscaling of coarse measurements. Section 7 is devoted to the estimation of scaling properties from data. Rather than covering data analysis in a comprehensive way (again, a downing task given the size limitations of the chapter), we choose to discuss four popular inference techniques, suggesting improvements and corrections. The fact that some popular procedures are inefficient or altogether incorrect is a significant problem, as many published results on scaling are consequently inaccurate or suspicious. Section 8 gives a brief overview of the use of scale invariance in four areas of hydrology, namely rainfall, river networks and fluvial erosion topography, river flow, and flow through saturated porous media. Concluding remarks are made in Section 9.

2. Fractal and Scale-invariant Sets

The objects studied by classical geometry have integer dimensions. For example, straight lines have dimension 1, planar figures, such as squares and triangles, have dimension 2, and cubes and other polyhedra in three-dimensional space have dimension 3. All these objects are locally smooth. By contrast, fractal geometry deals with sets that are highly irregular and have non-integer (fractal) dimension,[5] in the sense explained below. One example is one-dimensional Brownian motion, which gives the position of a particle that starts at the origin and, during constant increments of time, displaces by independent and identically distributed Gaussian amounts. Another example is the Sierpinski triangle, which is obtained by first dividing an equilateral triangle of side length *l* into four triangles of side length *l*/2 and removing the triangle at the center, and then repeating the same operation of subdivision and elimination on each of the remaining triangles of side length *l*/2, then the nine remaining triangles of side length *l*/4, and so on.

In order to be fractal, an object must not just be irregular, but the irregularities must in turn depend in a regular way on scale (such that the number of tiles needed to cover the object must be a power function of the tile size; see Section 2.1). This is why fractality often occurs concurrently with scale invariance, which, loosely speaking, is the property that, under transformations that involve a change of scale, any part of an object looks like the whole. For example, Brownian motion and the Sierpinski triangle are both fractal and scale-invariant objects. In spite of being often used interchangeably (including in the title of this chapter), fractality and scale invariance are not equivalent concepts. For example, a straight line on the (x, y) plane that passes through the origin reproduces itself under isotropic scaling and therefore is scale-invariant, but has dimension 1 and therefore is non-fractal. Our interest is in scale invariance, but since fractality is a frequent property of scale-invariant objects, we start with a brief review of fractal sets and their fractal dimensions.

2.1 *Fractal Sets and their Fractal Dimensions*

There are many definitions of fractal dimension. The most general and mathematically satisfactory one is the Hausdorff dimension, D_H. Its definition is rather technical[6] and is given below in a simplified form, which is sufficient in many cases, including scale-invariant objects.

Consider a set *S* of R^d and let $s_\delta \subset R^d$ be a measurable set, for example a segment in R^1; a square, a rectangle or a disc in R^2; etc. The set s_δ has diameter (maximum linear size) δ and area A_δ. Suppose that N_δ translated/rotated versions of s_δ are needed to cover *S*. Then the total area of the covering set is $N_\delta A_\delta$. If *S* has topological dimension less than *d*, for example a line in R^2 or R^3, then typically $\lim_{\delta\to 0}(N_\delta A_\delta) = 0$ (this is however not always the case; for example it is not for "space-