# A disaggregation model for storm hyetographs

**(Developed in the framework of the AFORISM project)**

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# Model requirements

## General characteristics:

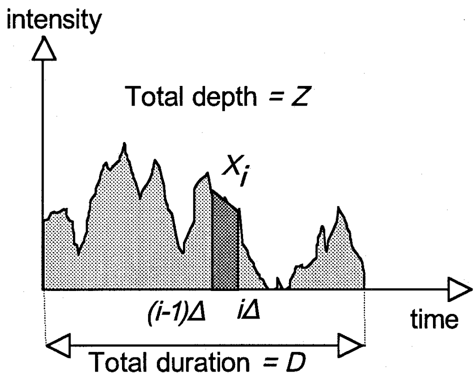
- Random shape of hyetograph
- Reproduction of the hyetograph at an incremental basis ($\Delta$)
- Compatibility with various rainfall models
- Simplicity
- Parsimony of parameters

## Special characteristics

- Event based
- Given duration, $D$
- Given total depth, $Z$

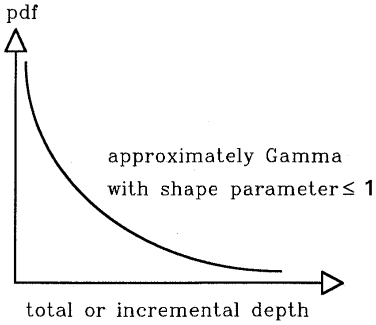
⇓

Disaggregation model



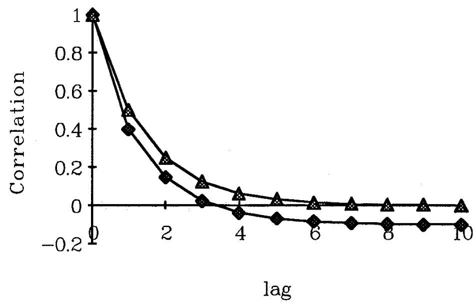
# General structure of total and incremental depths

## Marginal distribution




## Correlation structure

(incremental depths)






# Disaggregation models are not exact

## General form

$$\underbrace{X_1 + X_2 + \cdots + X_k}_{\text{lower level variables of } k \text{ sub-periods}} = \underbrace{Z}_{\text{higher level variable of one period}}$$

## General problems

- Skewed distributions
- Additive property (if applied to nonlinearly transformed variables)
- Correlations between variables located at different periods

## Exceptions, where d.m. are exact

1. $X_i$ are normal, independent of variables of other periods
2. $X_i$ are independent gamma variables with common scale parameter

# Theoretical background

## Let

- $X_i^*$: Independent, Gamma($\kappa_i$, $\lambda$) ($i$ = 1,...,$k$)
- $Z$: Independent of $X_i^*$, Gamma($\kappa$, $\lambda$), where $\kappa = \sum_{j=1}^{k} \kappa_j$
- $X_i = \dfrac{X_i^*}{\sum_{j=1}^{k} X_j^*} Z$

## Then

$X_i$ : Independent, Gamma($\kappa_i$, $\lambda$) ($i$ = 1,...,$k$)
(same as $X_i^*$)

Note: $X_i$'s add up to Z



# Theoretical background (continued)

## Summary of the proof

To proof that $X_i$ is independent of $X_j$ and both are gamma distributed:

1. Take the joint pdf of the variables (all are independent gamma distributed)

   $X_i^*$, $X_j^*$, $Y^* = X_1^* + \cdots + X_{i-1}^* + X_{i+1}^* + \cdots + X_{j-1}^* + X_{j+1}^* + \cdots + X_k^*$ and $Z$
2. Use the transformations

   $$X_i = \frac{X_i^*}{X_i^* + X_j^* + Y^*} Z,\ X_j = \frac{X_j^*}{X_i^* + X_j^* + Y^*} Z,\ Y = \frac{X_i^*}{X_i^* + X_j^* + Y^*} Z \text{ and}$$

   $W = X_i^* + X_j^* + Y^*$
3. Determine the joint pdf of ($X_i$, $X_j$, $Y$, $W$)
4. Obtain that ($X_i$, $X_j$, $Y$, $W$) are independent and gamma distributed

# Practical considerations

## Model course

1. Get the known $Z$.
2. By using a sequential model determined by $X_i$'s, generate $X_i^*$'s independently of $Z$.
3. By using the above transformations obtain $X_i$'s.

## Limitations

In strict sense:

- $X_i$: 2-parameter gamma distributed
- $E[X_i]/Var[X_i] = ct$ (independent of $i$)
- $X_i$ independent of $X_j$

In non-strict sense (approximations):

- $X_i$: 3-parameter gamma distributed
- $E[X_i]/Var[X_i] \approx E[X_j]/Var[X_j]$
- $Corr[X_i, X_j]$: not too large



# The sequential model

## Overview

$$\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{bmatrix} = \begin{bmatrix} \omega_{11} & 0 & \cdots & 0 \\ \omega_{21} & \omega_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{k1} & \omega_{k2} & \cdots & \omega_{kk} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_k \end{bmatrix}$$ or $\boldsymbol{X} = \boldsymbol{\Omega V}$ ($V_i$ independent, appr. 3-par. gamma)

## Parameter estimation

$\Omega\Omega^T = Cov[\boldsymbol{X}, \boldsymbol{X}] \Rightarrow \Omega$ by decomposition(lower triangular)

$\omega_{ii} E[V_i] = E[X_i] - \sum_{l=1}^{i-1} \omega_{il} E[V_l]$

$Var[V_i] = 1$

$\omega_{ii}^3 \mu_3[V_i] = \mu_3[X_i] - \sum_{l=1}^{i-1} \omega_{il}^3 \mu_3[V_l]$

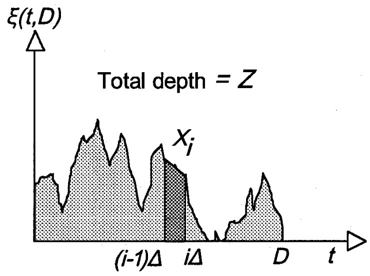
## Link to a rainfall model

Inputs to the sequential model: $E[\boldsymbol{X}]$, $Cov[\boldsymbol{X}, \boldsymbol{X}]$, $\mu_3[\boldsymbol{X}]$

More inputs for disaggregation: $E[Z]$, $Var[Z]$, $\mu_3[Z]$

# Application 1 - Scaling rainfall model

## General structure




$$\{\xi(t,D)\} \overset{d}{=} \{\lambda^{-H}\xi(\lambda t,\lambda D)\}$$

where $\xi()$: instantaneous rainfall intensity
D: duration of the event
t: time ($0 \leq t \leq D$)
H: scaling exponent

$$E[\xi(t,D)] = c_1 D^H$$
$$Cov[\xi(t,D)\xi(t+\tau,D)] = \left(\varphi(\tau / D) - c_1^2\right)D^{2H}$$
$$\varphi(y) = \tfrac{1}{2}(c_2 + c_1^2)(1-\beta)(2-\beta)\ y^{-\beta}$$



# Application 1 - Scaling rainfall model (continued)

## Statistics of incremental and total depths

$$\mathrm{E}[X_i] = c_1 \delta D^{H+1}$$
$$\mathrm{Var}[X_i] = \left[(c_2 + c_1^2)\delta^{-\beta} - c_1^2\right]\delta^2 D^{2(H+1)}$$
$$\mathrm{Cov}[X_i, X_j] = \left[(c_2 + c_1^2)\delta^{-\beta} f(|j-i|,\beta) - c_1^2\right]\delta^2 D^{2(H+1)}$$
$$\mathrm{E}[Z] = c_1 D^{H+1}$$
$$\mathrm{Var}[Z] = c_2 D^{2(H+1)}$$

where

$$\delta = \Delta / D$$
$$f(m,\beta) = \tfrac{1}{2}\left[(m-1)^{2-\beta} + (m+1)^{2-\beta}\right] - m^{2-\beta}, \quad m > 0$$

## Parameters

$c_1$ (= 1.05): mean value parameter
$c_2$ (= 0.44): variance parameter
$\beta$ (= 0.32): correlation decay parameter
$H$ (= -0.20): scaling exponent

# Application 2 - ARMA(1,1) model

## General structure

Stationary model, Markovian in continuous time: $\mathrm{Cov}[\xi(t), \xi(t+\tau)] = k_2 e^{-\beta_2 \tau}$

## Statistics of incremental and total depths

$$
\begin{aligned}
&\mathrm{E}[X_i] = c_1 \Delta \\
&\mathrm{Var}[X_i] = 2(k_2^2 / \beta_2^2)(\beta_2 \Delta - 1 + e^{-\beta_2 \Delta}) \\
&\mathrm{Cov}[X_i, X_j] = 2(k_2^2 / \beta_2^2)(1 - e^{-\beta_2 \Delta})^2 e^{-\beta_2(|j-i|-1)\Delta} \\
&\mathrm{E}[Z] = c_1 D \\
&\mathrm{Var}[Z] = 2(k_2^2 / \beta_2^2)(\beta_2 D - 1 + e^{-\beta_2 D})
\end{aligned}
$$

## Parameters

| | | |
|---|---|---|
| $c_1$ | (=0.65): | mean value parameter |
| $k_2$ | (=1.25): | variance parameter |
| $\beta_2$ | (=1.58): | correlation decay parameter |



# Application 3 - AR(1) model

## General structure

Stationary model, Markovian in discrete time: $\mathrm{Cov}[X_i, X_j] = C\, \rho^{|i-j|}$

## Parameters = statistics of incremental depths

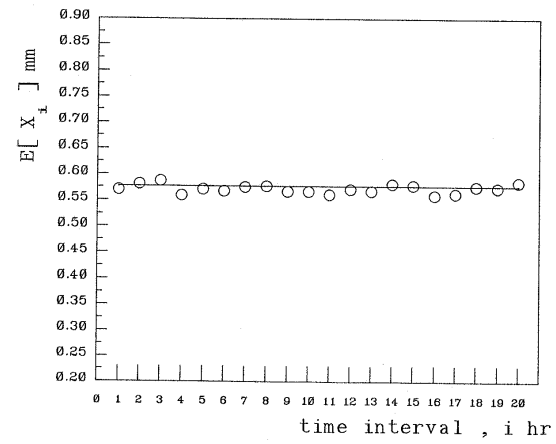
| | | |
|---|---|---|
| E[Xi]: | (=0.577) | mean of incremental depth (for D = 1 h) |
| C = Var[Xi] | (=0.776) | variance of incremental depth (for D = 1 h) |
| $\rho$ | (=0.53) | first autocorrelation coefficient |

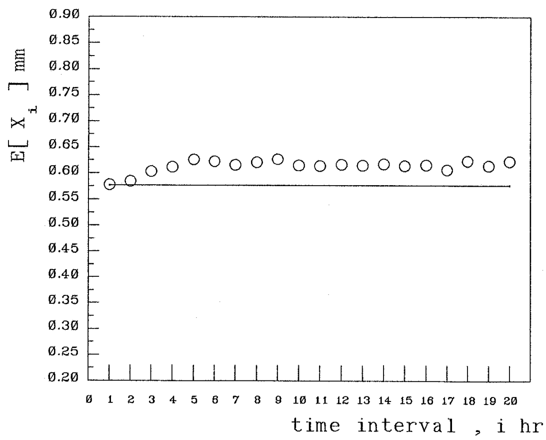
## Statistics of total depth

$$
\begin{aligned}
&\mathrm{E}[Z] = k\,\mathrm{E}[X_i] \\
&\mathrm{Var}[Z] = \textstyle\sum_{i=1}^{k} \sum_{j=1}^{k} \mathrm{Cov}[X_i, X_j]
\end{aligned}
$$

# Simulation results - Application 1 - Scaling

## Mean values (Disaggregation and sequential model)

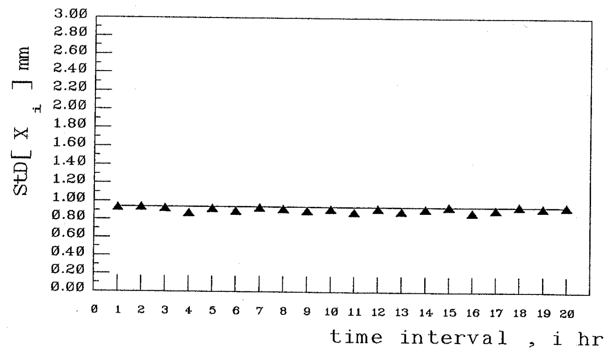


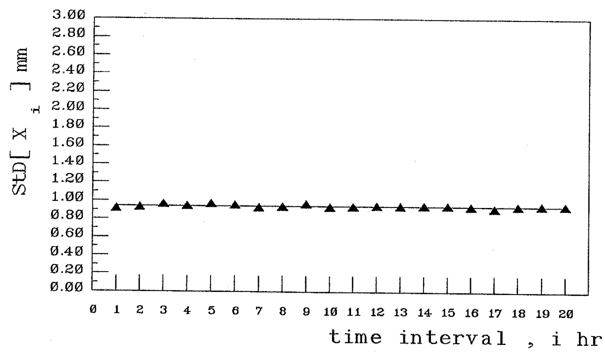






# Simulation results - Application 1 - Scaling

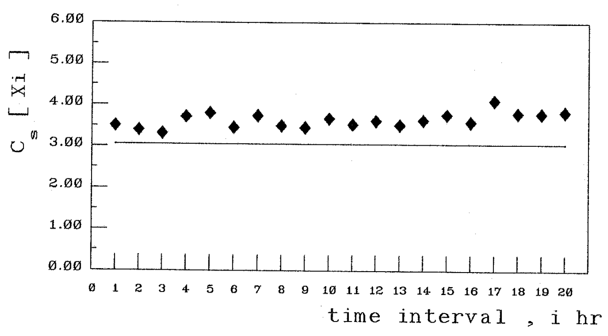
## Standard deviations (Disaggregation and sequential model)

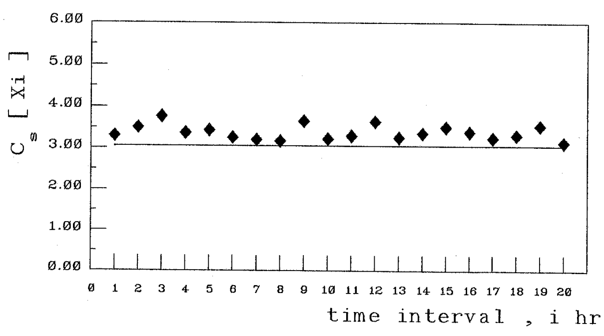


# Simulation results - Application 1 - Scaling

## Skewness coefficients (Disaggregation and sequential model)

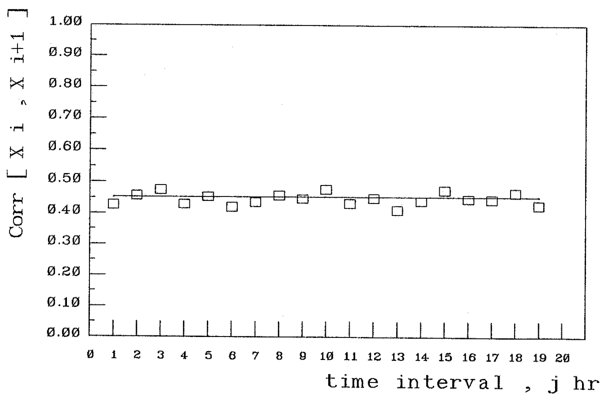


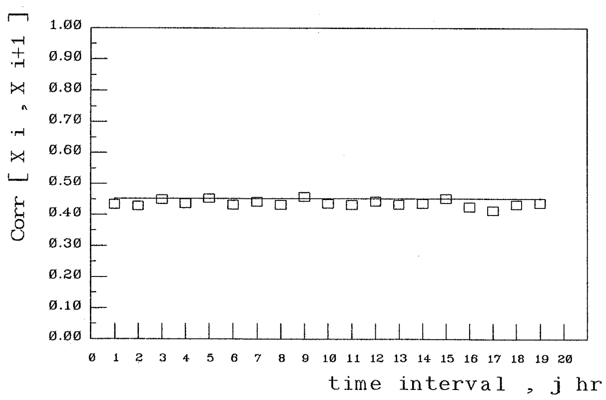






# Simulation results - Application 1 - Scaling

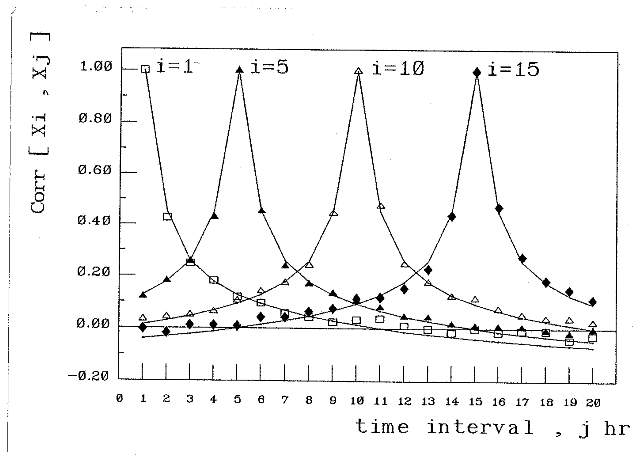
## First autocorrelation coefficients (Disaggregation and sequential model)

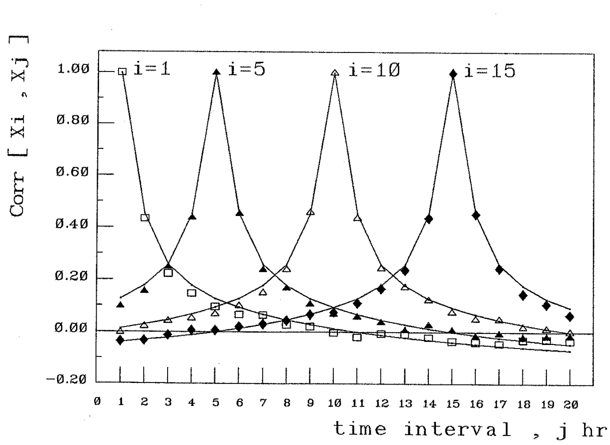


# Simulation results - Application 1 - Scaling

## Autocorrelation coefficients (Disaggregation and sequential model)

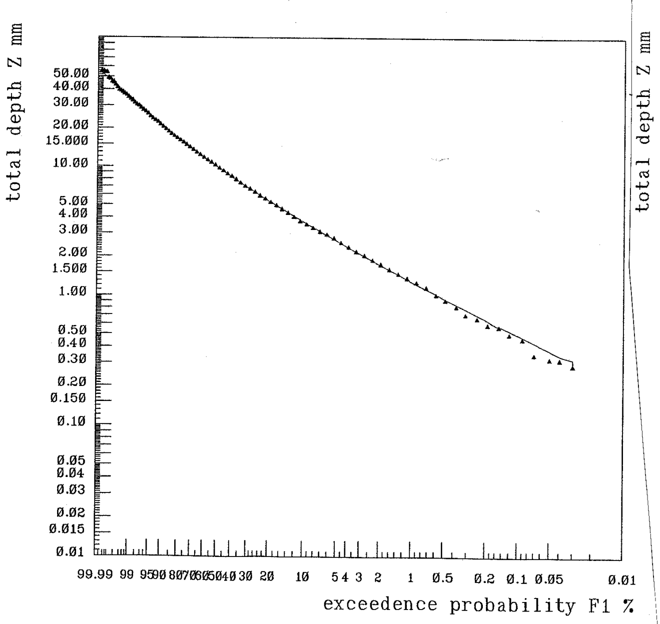


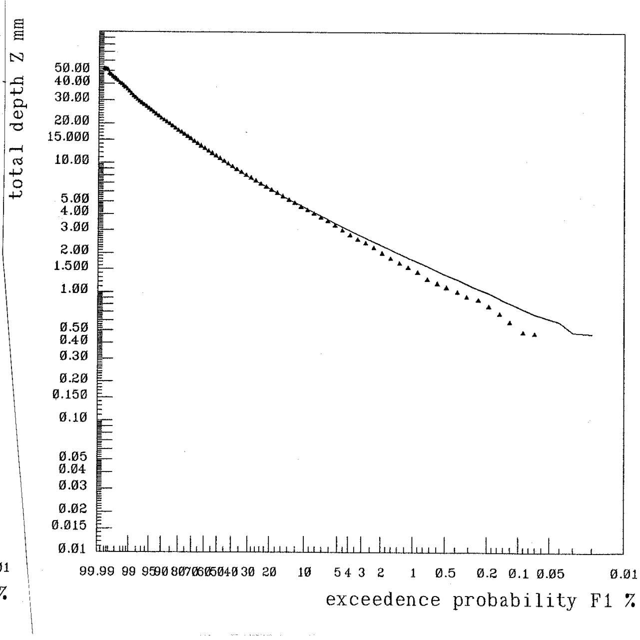






# Simulation results - Application 1 - Scaling

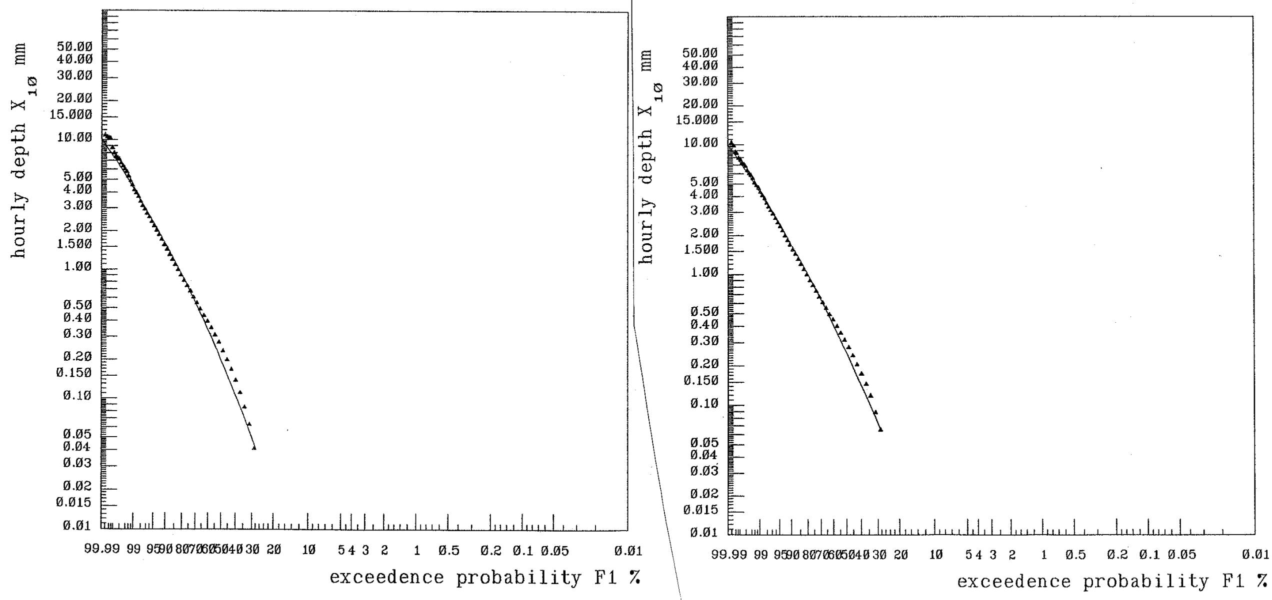
## Distribution function of *Z* (Disaggregation and sequential model)



# Simulation results - Application 1 - Scaling

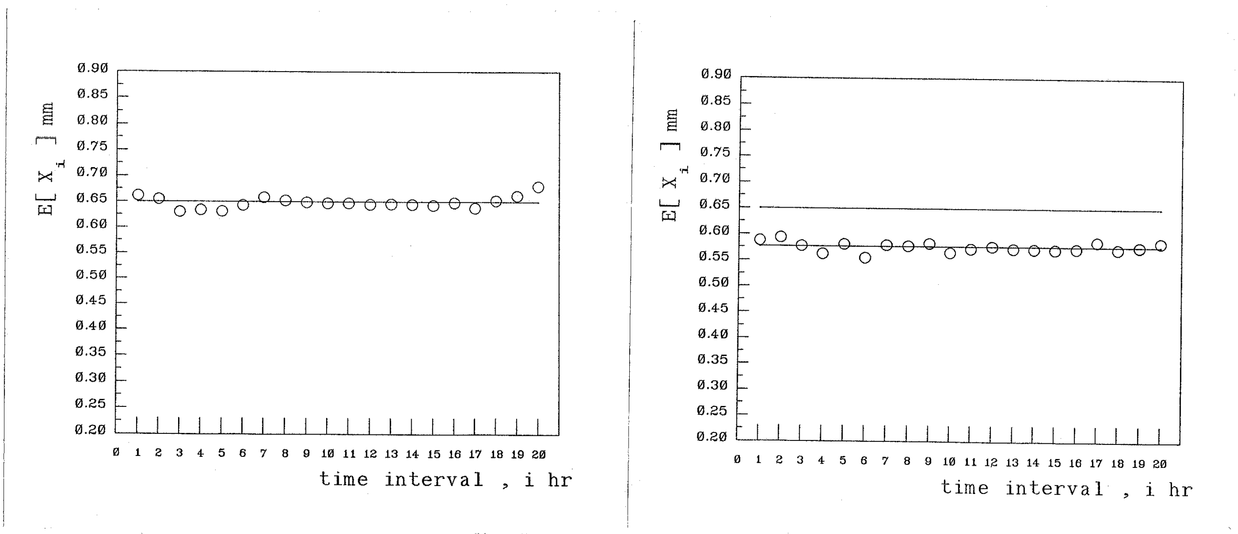
## Distribution function of $X_{10}$ (Disaggregation and sequential model)





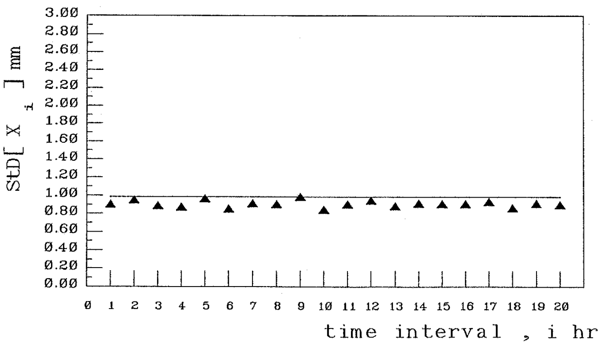

# Simulation results - Application 2 - ARMA(1,1)

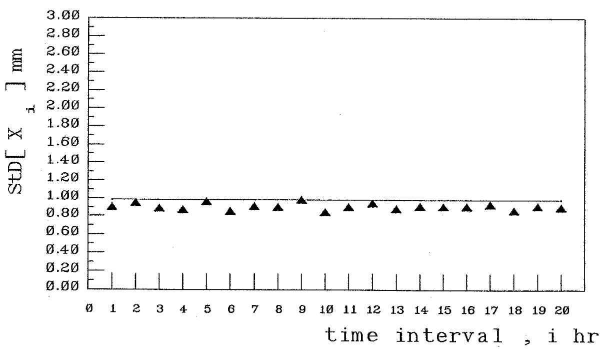
## Mean values (Consistent and inconsistent case)



# Simulation results - Application 2 - ARMA(1,1)

## Standard deviations (Consistent and inconsistent case)

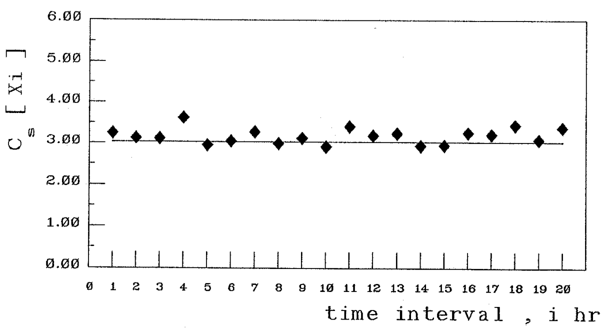


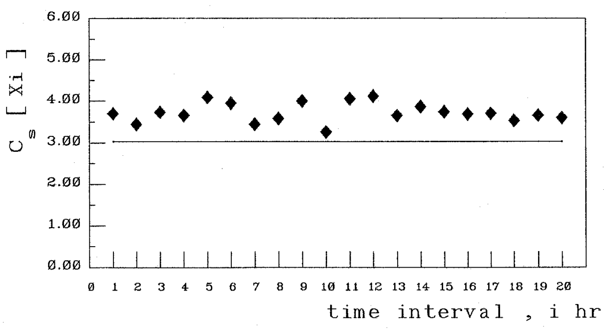






# Simulation results - Application 2 - ARMA(1,1)

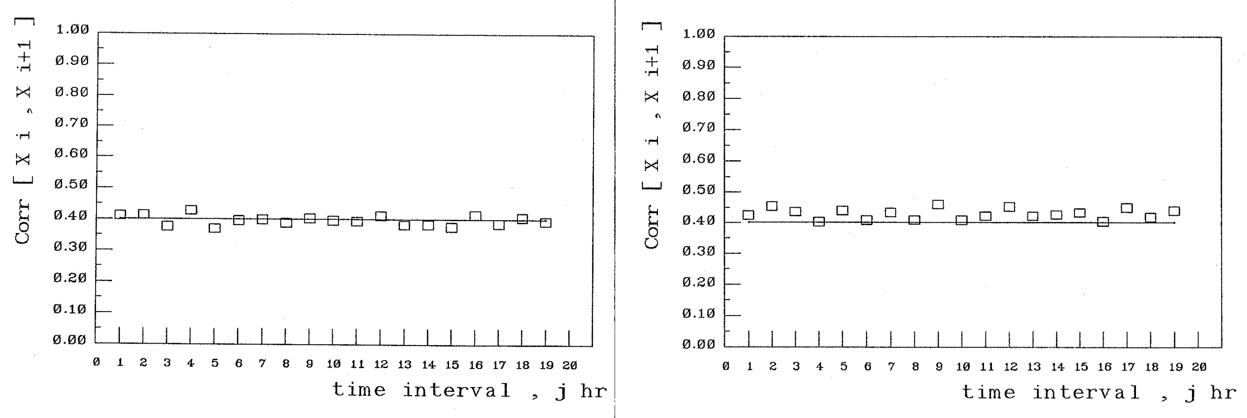
## Skewness coefficients (Consistent and inconsistent case)



# Simulation results - Application 2 - ARMA(1,1)

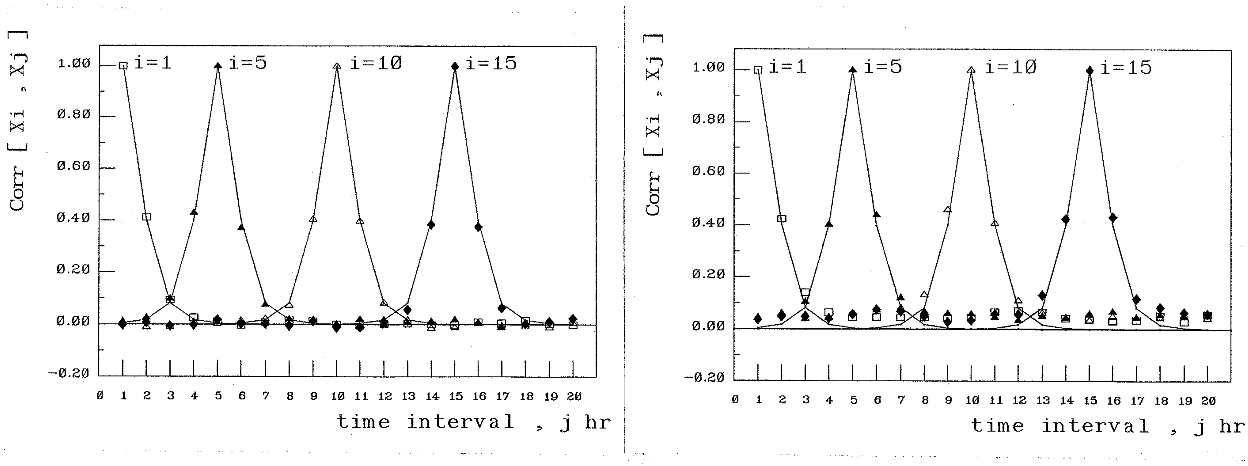
## First autocorrelation coefficients (Consistent and inconsistent case)





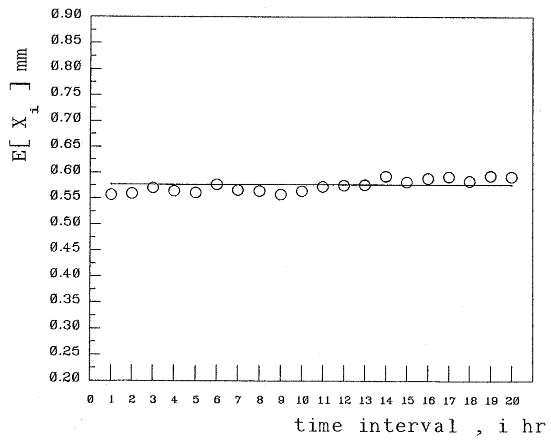

# Simulation results - Application 2 - ARMA(1,1)

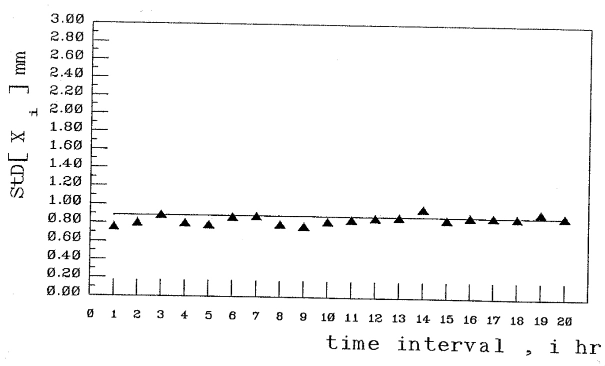
## Autocorrelation coefficients (Consistent and inconsistent case)



# Simulation results - Application 3 - AR(1)

## Mean values and Standard deviations

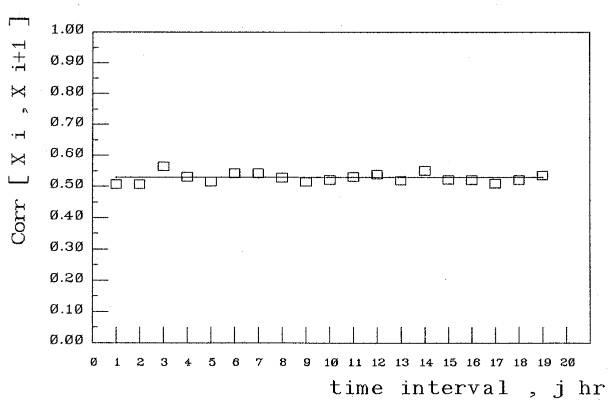


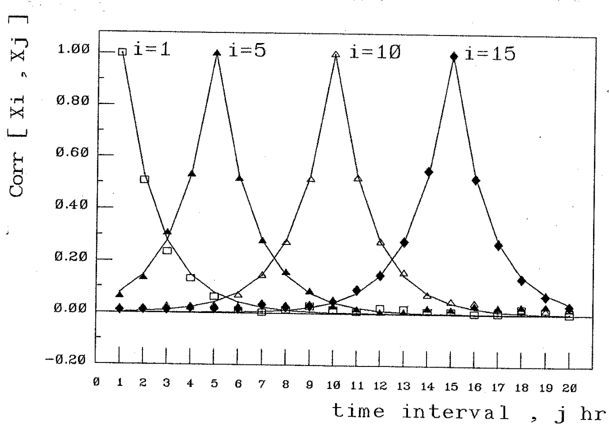






# Simulation results - Application 3 - AR(1)

## Autocorrelation coefficients



# Simulation results - Application 1 - Scaling

## Dimensionless mass curves