A quick gap-filling of missing hydrometeorological data

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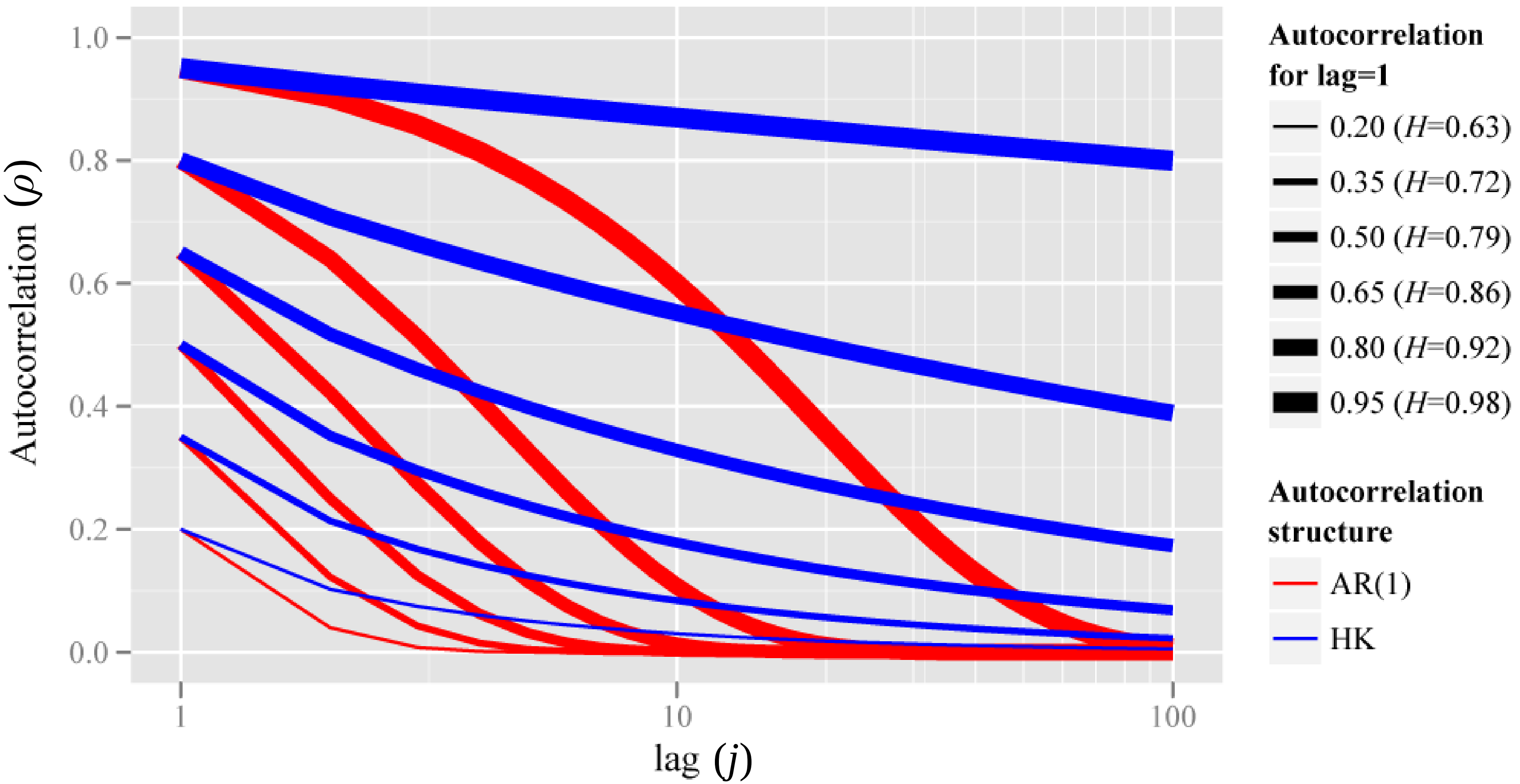


Exponential ACS : $\rho_j = \rho^{|j|}$

Power-type ACS : $\rho_j = \frac{1}{2}\left[\left(j+1\right)^{2H} + \left(j-1\right)^{2H}\right] - j^{2H}$

- Given that **2×N observations** are available, we want to estimate a **missing value y** :

$$x_{-N}, \ldots, x_{-1}, \boxed{y,} \; x_1, \ldots, x_N$$

- A (linear) **estimate of y** can be expressed as:

$$\underline{y} = w_{-N} x_{-N} + \ldots + w_N x_N + \underline{e}$$

where

x_i : the observed values

w_i : weighting factors

$\underline{e}$: estimation error

- The **Mean Squared Error** of the estimation is then defined as:

$$\mathrm{MSE} := \mathrm{E}\left[e^2\right] = \mathrm{E}\left[\left(y - \underline{y}\right)^2\right]$$

- We examine the following **estimate for *y***:

$$\underline{y} = \frac{\sum_{i=1}^{n} x_{-i} + \sum_{i=1}^{n} x_i}{2n}$$

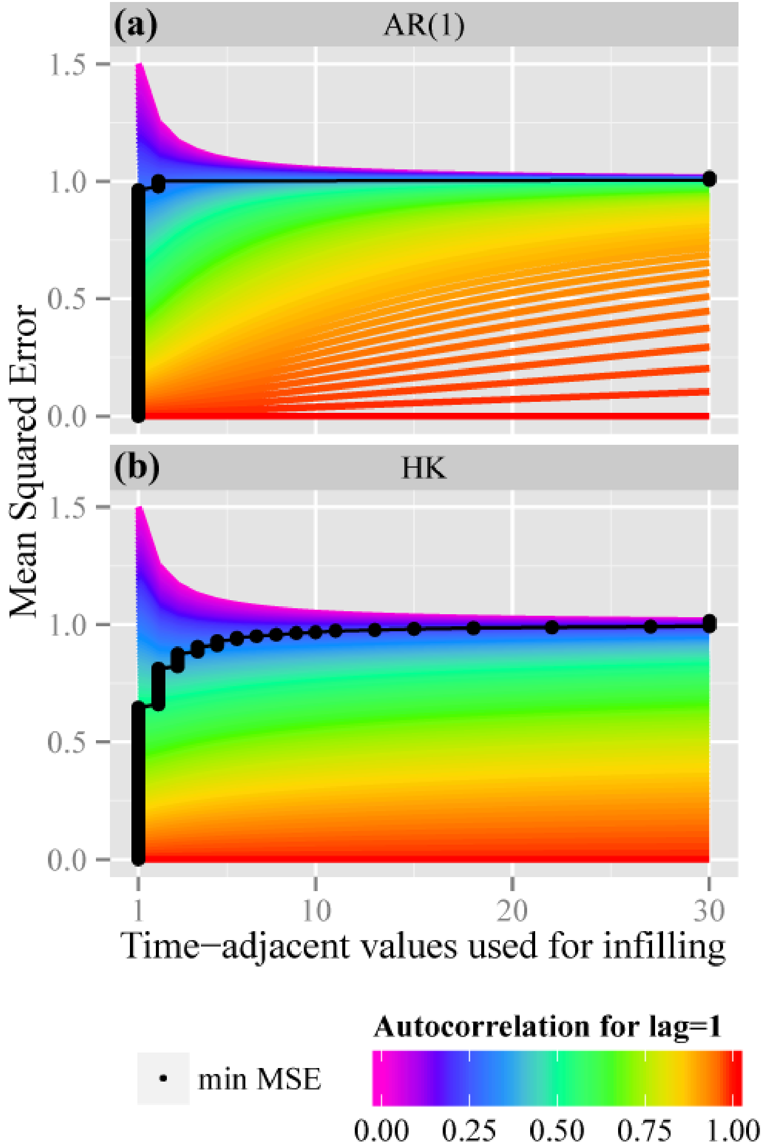
$$x_{-N}, \ldots, x_{-n}, \ldots, x_{-1}, y, x_1, \ldots, x_n, \ldots, x_N$$

- Assuming that the underlying process is (weakly) **stationary**, the **MSE** of the estimation is given by:

$$\text{MSE} := \text{E}\left[e^2\right] = \text{E}\left[\left(y - \underline{y}\right)^2\right] = \text{E}\left[\left(y - \frac{\sum_{i=1}^{n} x_{-i} + \sum_{i=1}^{n} x_i}{2n}\right)^2\right]$$

$$= \frac{1}{2}\left(\frac{\sigma}{n}\right)^2 \left[(2n+1)\left(n - 2\sum_{i=1}^{n} \rho_i\right) + \sum_{i=1}^{2n} (2n+1-i)\,\rho_i\right]$$

- Which is the **optimal** (i.e., minMSE) number of **neighbouring values (*n*)** that should be used**?**



$$\underline{y} = \frac{\sum_{i=1}^{n} x_{-i} + \sum_{i=1}^{n} x_i}{2n}$$

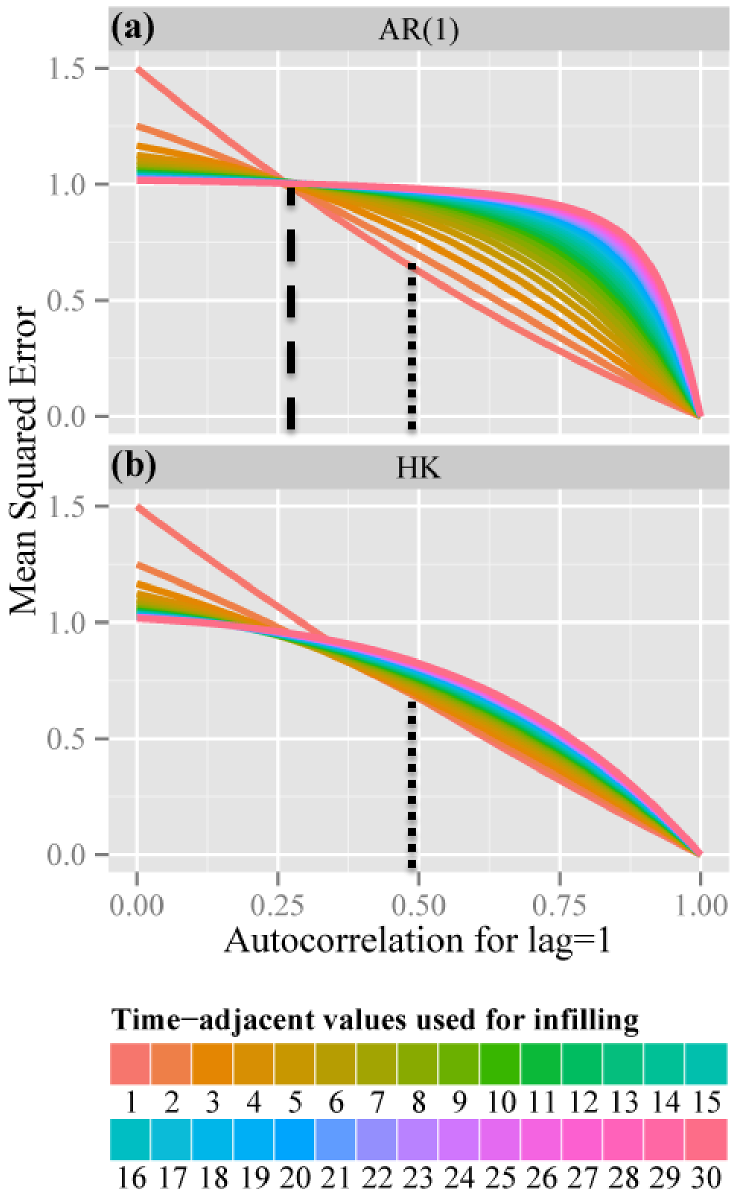
- **AR(1)**

For a wide range of **lag-1 autocorrelations**, the strictly local average (i.e., ***n*=1**) provides the **minMSE**.

- **HK**

As the **lag-1 autocorrelation increases**, the time-adjacent values (***n***) required for a **minMSE** gradually **decrease**.

See also
Dialynas et al. (2010)
http://itia.ntua.gr/en/docinfo/981/
Pappas (2010)
http://itia.ntua.gr/en/docinfo/1065/



$$\underline{y} = \frac{\sum_{i=1}^{n} x_{-i} + \sum_{i=1}^{n} x_i}{2n}$$

Markovian property:
"The future does not depend on the past when the present is known" [*Papoulis*, 1965, p.535].

Optimal Local Average			
Short-term persistence -AR(1)-		Long-term persistence -HK-	
$\rho \leq 0.25$	n=$n_{\max}$	$\rho \leq 0.3$	n=$n_{\max}$
$0.25 < \rho \leq 0.28$	n=2	$0.30 < \rho \leq 0.32$	n=4
		$0.32 < \rho \leq 0.38$	n=3
$\rho > 0.28$	n=1	$0.38 < \rho \leq 0.51$	n=2
		$\rho > 0.51$	n=1

ρ: lag-one autocorrelation coefficient
n: time-adjacent values used for the infilling
n_{max}: all the available observed values, i.e., total/sample average

For **both ACS** (exponential or power-type) when $\boldsymbol{\rho}$**>0.51** the strictly local average (**n=1**) provides the **minMSE**.

- **Generalization** of the OLA methodology, so that **information** from both **local** and **global average** will be used according to the **lag-1 autocorrelation**.

- We examine the following **estimate for *y***:

$$\underline{y} = \lambda \frac{\sum_{i=-N}^{N} x_i}{2N} + (1-\lambda)\frac{x_{-1}+x_1}{2}$$

Total (sample) average — **Local** (strictly) average

where λ **is the weighting factor** for the **total (sample) average** and the **local (strictly) average**.

- Parameter λ reflects the strength of the temporal autocorrelation:

low values → **high** correlation
high values → **low** correlation

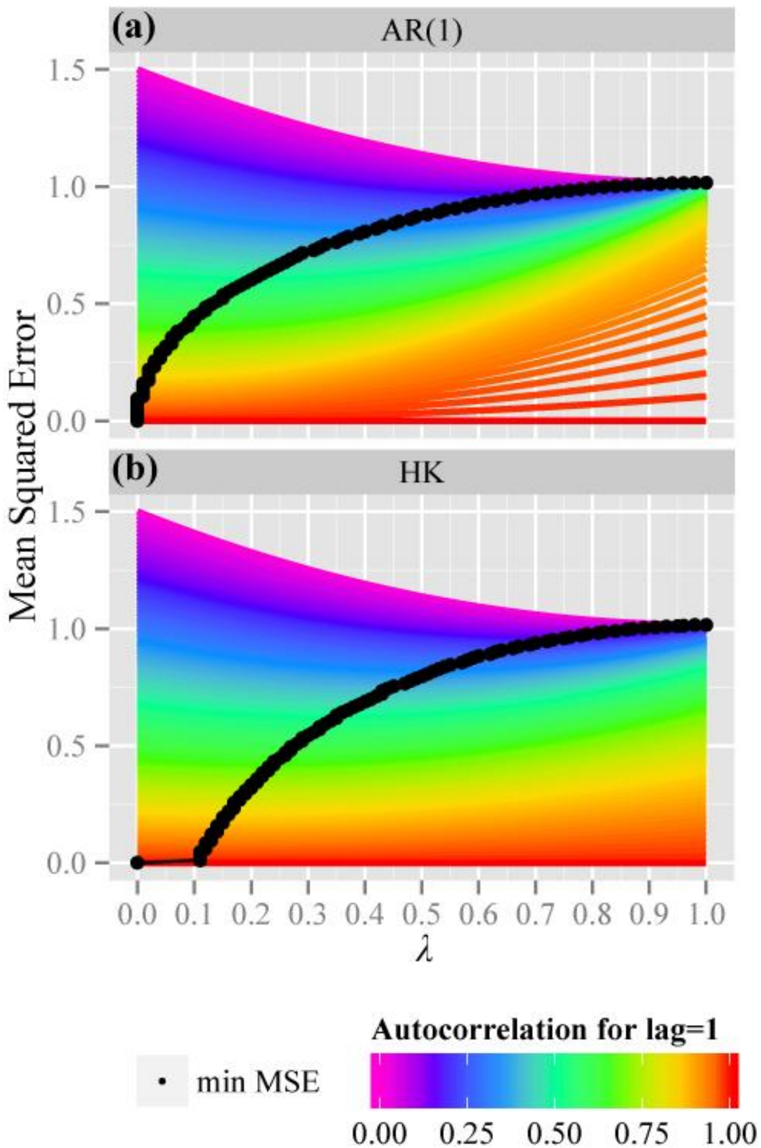
See also:
Pappas (2010)
http://itia.ntua.gr/en/docinfo/1065/

- Assuming that the underlying process is (weakly) **stationary**, the **MSE** of the estimation is then defined as:

$$\text{MSE} := \text{E}\left[e^2\right] = \text{E}\left[\left(y - \underline{y}\right)^2\right] = \text{E}\left[\left(y - \left(\lambda \frac{\sum_{i=-N}^{N} x_i}{2N} + (1-\lambda)\frac{x_{-1} + x_1}{2}\right)\right)^2\right]$$

- After some **algebraic** manipulations:

$$\text{MSE} = \frac{1}{2}\sigma^2(3 - 4\rho_1 + \rho_2) - 2\lambda\sigma^2\left[\frac{1}{N}\sum_{i=1}^{N}\rho_i - \frac{1}{2N}\left(\sum_{i=1}^{N-1}\rho_i - \sum_{i=2}^{N+1}\rho_i + 1\right) - \rho_1 + \frac{\rho_2}{2} + 0.5\right]$$
$$+\lambda^2\sigma^2\left[\frac{1}{2N^2}\left(2\sum_{i=1}^{N-1}(N-i)\rho_i + \sum_{i=2}^{N+1}(i-1)\rho_i + \sum_{i=N+2}^{2N}(2N+1-i)\rho_i + N\right) + \frac{\rho_2}{2} + \frac{1}{2} - \frac{1}{N}\left(\sum_{i=1}^{N-1}\rho_i + \sum_{i=2}^{N+1}\rho_i + 1\right)\right]$$



$$\underline{y} = \lambda \frac{\sum_{i=-N}^{N} x_i}{2N} + (1-\lambda)\frac{x_{-1} + x_1}{2}$$

- **AR(1) & HK**

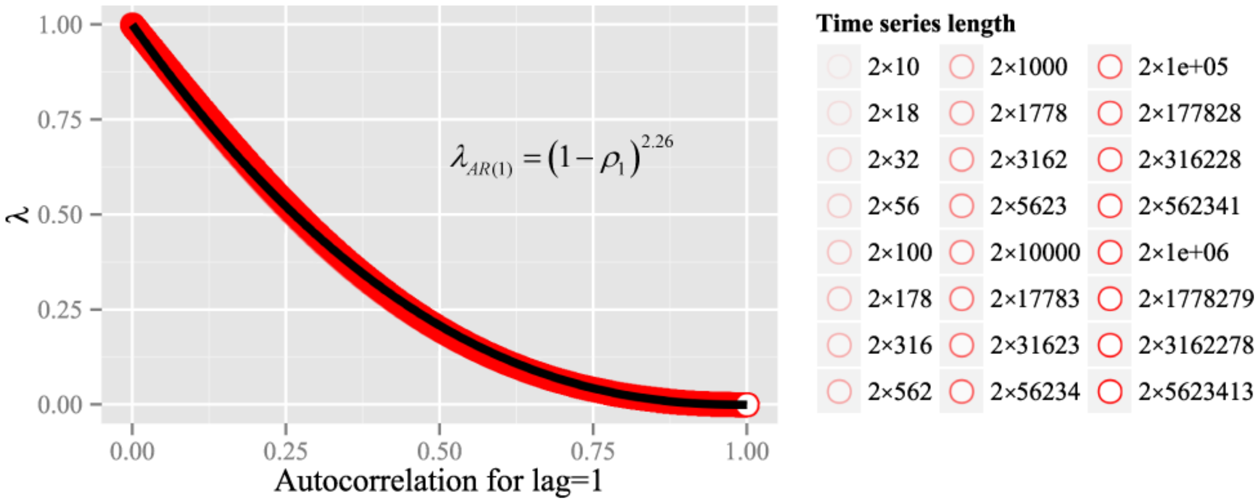
As **lag-1 autocorrelation** increases, the contribution of the local average increases (i.e., lower values of $\boldsymbol{\lambda}$).

- **HK**

It *takes time* for the **HK process** to **reveal** its **properties**.

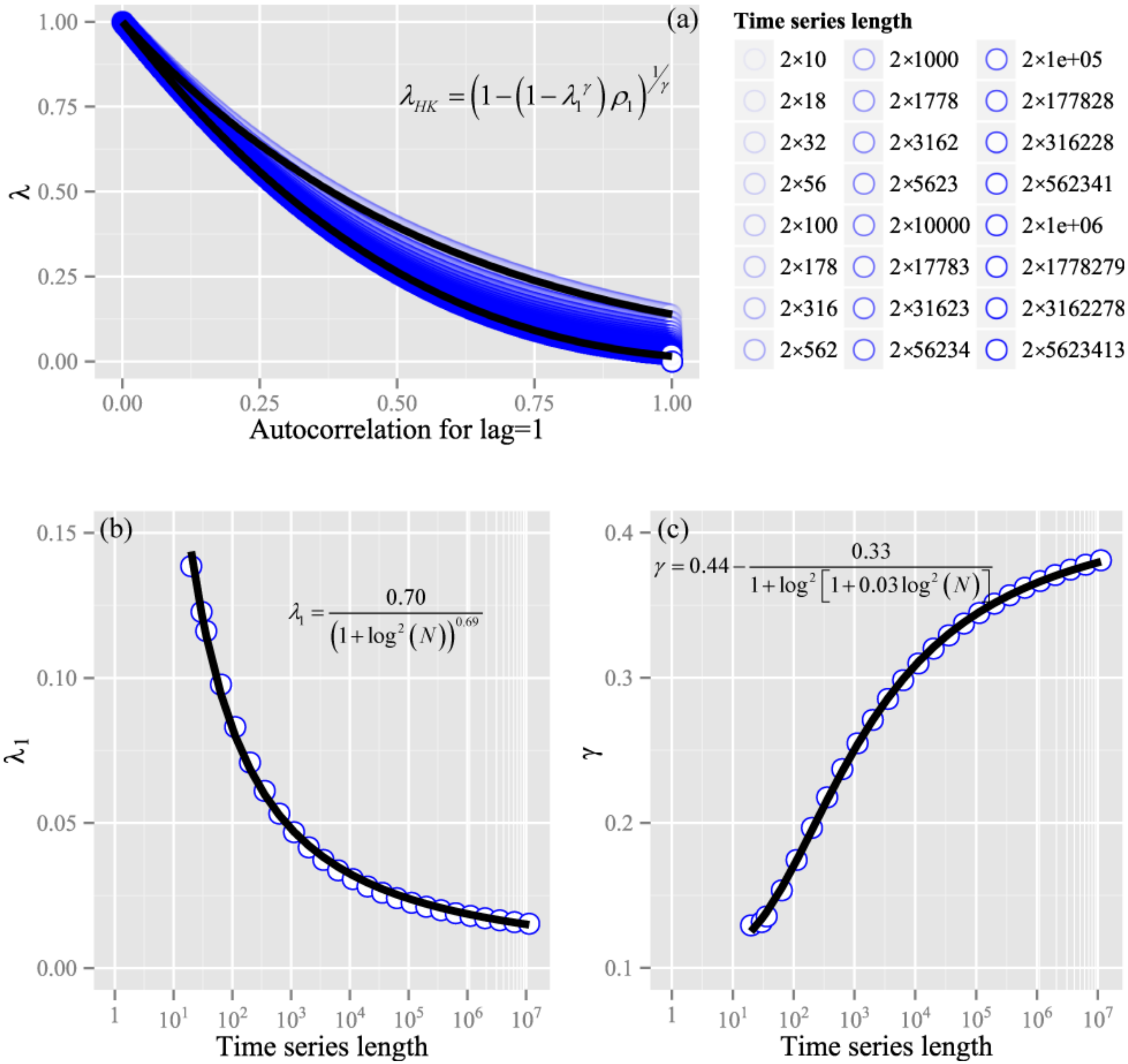
- The influence of **sample size (N)** :

$$\mathrm{MSE} = \frac{1}{2}\sigma^2(3-4\rho_1+\rho_2) - 2\lambda\sigma^2\left[\frac{1}{N}\sum_{i=1}^{N}\rho_i - \frac{1}{2N}\left(\sum_{i=1}^{N-1}\rho_i - \sum_{i=2}^{N+1}\rho_i + 1\right) - \rho_1 + \frac{\rho_2}{2} + 0.5\right]$$
$$+\lambda^2\sigma^2\left[\begin{array}{l}\frac{1}{2N^2}\left(2\sum_{i=1}^{N-1}(N-i)\rho_i + \sum_{i=2}^{N+1}(i-1)\rho_i + \sum_{i=N+2}^{2N}(2N+1-i)\rho_i + N\right) \\ +\frac{\rho_2}{2} + \frac{1}{2} - \frac{1}{N}\left(\sum_{i=1}^{N-1}\rho_i + \sum_{i=2}^{N+1}\rho_i + 1\right)\end{array}\right]$$



$$\underline{y} = \lambda_{AR(1)} \frac{\sum_{i=-N}^{N} x_i}{2N} + (1-\lambda_{AR(1)}) \frac{x_{-1} + x_1}{2}$$

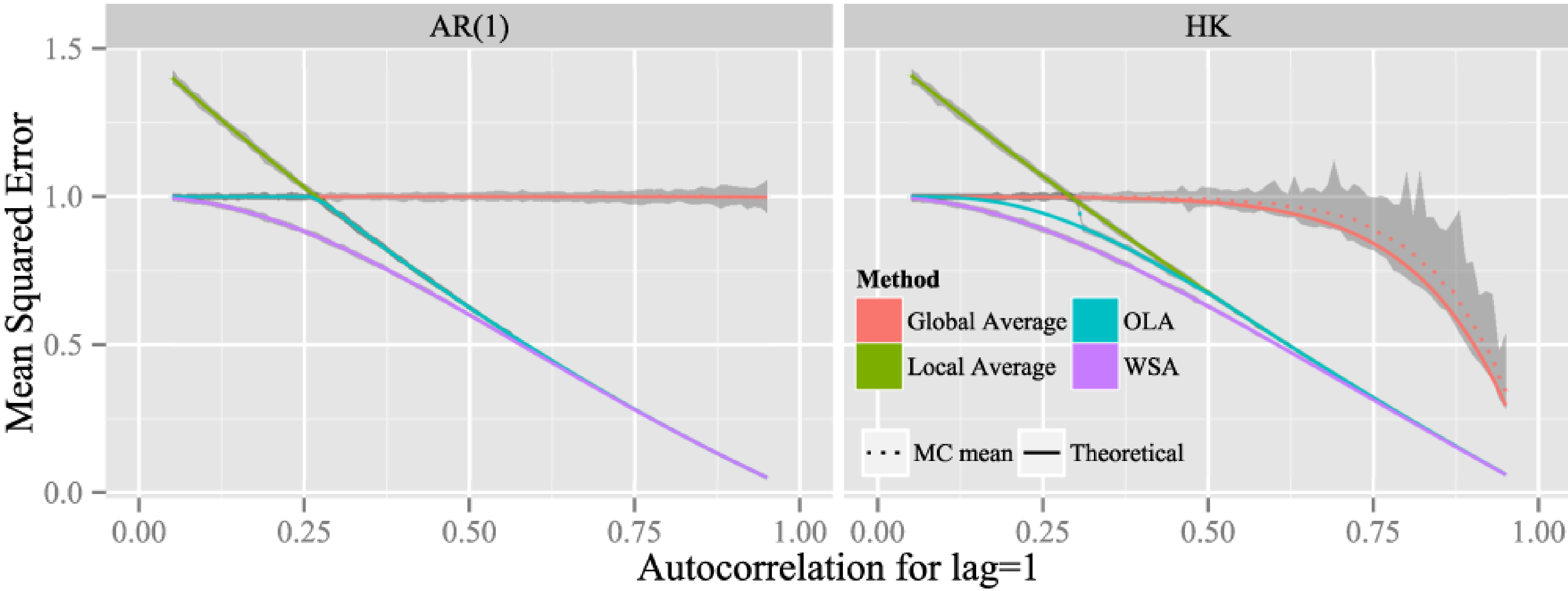
- For the case of **exponential ACS**, the $\boldsymbol{\lambda}$-$\boldsymbol{\rho_1}$ relationship does **not vary** significantly with time series length $\boldsymbol{N}$.



$$\underline{y} = \lambda_{HK}\frac{\sum_{i=-N}^{N} x_i}{2N} + (1-\lambda_{HK})\frac{x_{-1}+x_1}{2}$$

- For the case of **power-type ACS**, the $\boldsymbol{\lambda}$-$\boldsymbol{\rho_1}$ relationship **varies** significantly with time series length $\boldsymbol{N}$.

- To circumvent this issue, the $\boldsymbol{\lambda}$-$\boldsymbol{\rho_1}$ is approximated using two additional parameters ($\boldsymbol{\lambda_1}$, $\boldsymbol{\gamma}$).


AR(1)
HK
1.5
1.0
0.5
0.0
Mean Squared Error
0.00
0.25
0.50
0.75
1.00
Autocorrelation for lag=1
Method
Global Average
Local Average
OLA
WSA
MC mean
Theoretical

- We provide a **definitive argument against** the effortless use of **global (sample) mean** for infilling hydrometeorological (i.e., correlated) data.

- Local average ($\boldsymbol{n=1}$) is preferable for:
  - **Markovian** processes with $\boldsymbol{\rho > 0.28}$
  - **HK** processes with $\boldsymbol{\rho > 0.51}$

Tobler's first law in geography:
"Everything is related to everything else, but near things are more related than distant things" [*Tobler*, 1970]

- A generalized framework, described by the **Weighted Sum of local and total Average** (WSA), is developed and its advantages are demonstrated.

- The **WSA** methodology is therefore tailored for a **quick** infilling of **sporadic gaps** in hydrometeorological time series.

- Dialynas, Y., P. Kossieris, K. Kyriakidis, A. Lykou, Y. Markonis, C. Pappas, S.M. Papalexiou, and D. Koutsoyiannis, Optimal infilling of missing values in hydrometeorological time series, *European Geosciences Union General Assembly 2010, Geophysical Research Abstracts, Vol. 12*, Vienna, EGU2010-9702, European Geosciences Union, 2010.

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Many thanks to…

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Thank you!

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