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Assessing the error of geometry-based discretizations in groundwater modelling

E. Rozos and D. Koutsoyiannis, National Technical University of Athens, Department of Water Resources and Environmental Engineering, Athens, Greece

Introduction

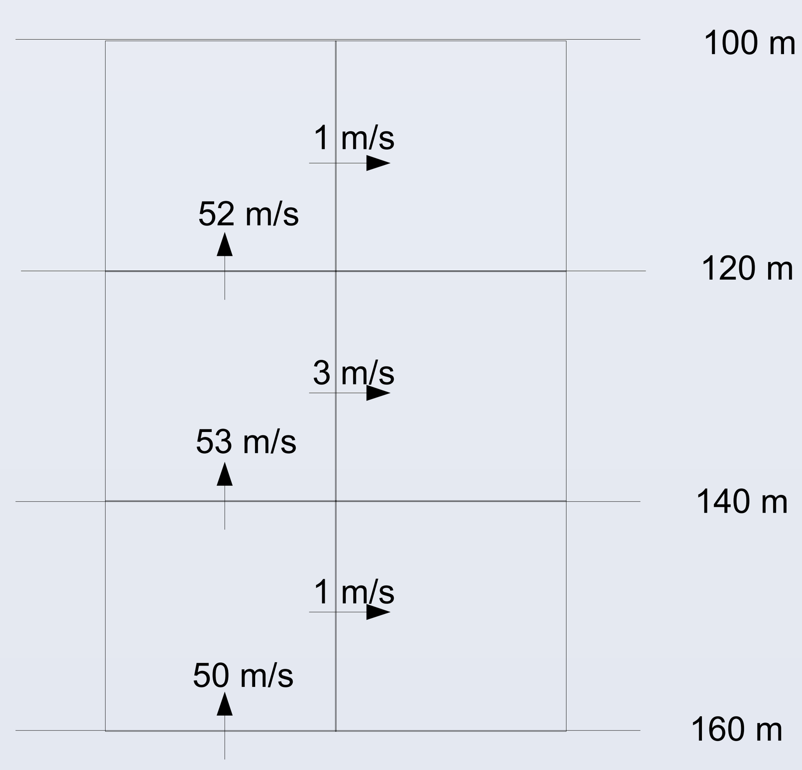
The dominant numerical methods for solving partial differential equations, pertaining to groundwater problems, are the Finite Difference Method (FDM), the Finite Element Method (FEM) and the Finite Volume Method (FVM). All these methods rely on a discretization of the flow domain that is guided by the boundary conditions and the locations of interest (measurements, pumps, etc). The disadvantages of these methods are that the discretization of the FDM is not very flexible whereas the other two have quite complicated mathematics. Rozos and Koutsoyiannis (2010) suggested the use of a multi-cell modelling approach that discretizes the flow domain based on its geometry (i.e. the flow lines and equipotential lines). This concept is more or less equivalent to the flow-nets, which have been introduced since the beginning of 20th century by Philipp Forchheimer to calculate the leakages under dams (Ettema, 2006). The advantages of this approach are that the discretization can be accomplished using a small number of irregularly shaped cells and that this approach results in simple algebraic equations. This approach is called Finite Volume Method with Simplified Integration (FVMSI) because it is a simplification of the FVM.

In a FVMSI mesh, the cells' boundaries should be either equipotential or flow lines (1st FVMSI condition). Consequently, all cells between two successive equipotential lines (a row of cells) should have similar simulated hydraulic heads and hence only minimal flux should take place between them (lateral flux). However, because of modelling errors, generally this will not be the case. If there are significant lateral fluxes, then the solution per se manifests an inconsistency of the mesh. In other words, since the solution indicates significant flux between some cells of the same row, then these cells should have been arranged into different rows (i.e., the mesh design is flawed).

Methods

It is suggested that the significance of a lateral flux can be measured by comparing it against the corresponding longitudinal flux (the flux between two adjacent cells having an equipotential line as a common edge). High significances are expected in parts of the mesh that do not comply with the FVMSI conditions.

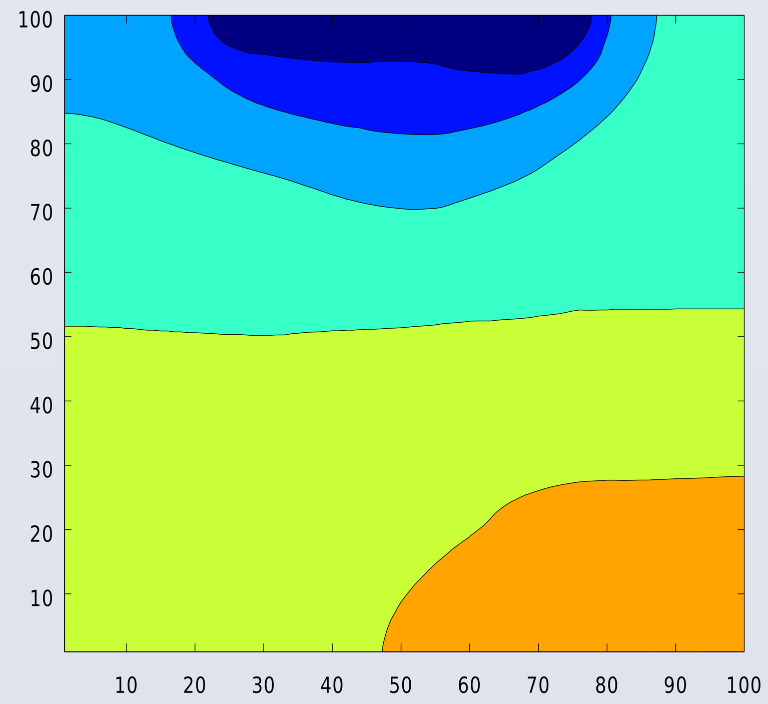
For example, the arrows in the figure next represent the fluxes, and the numbers the flux magnitudes. If this aquifer is isotropic and homogeneous, then the ratios 1/50, 3/53 and 1/52 are the significances of these lateral fluxes. In case the aquifer is not isotropic, it can be transformed to isotropic using the appropriate mapping (Strack, 1999). In case the aquifer is not homogeneous then, before obtaining the ratios, the fluxes should be divided by the corresponding lateral and longitudinal trans-boundary conductivities.



Example of lateral and longitudinal fluxes.

The reliability of the ratio of the fluxes as an indicator of the compliance of a mesh was evaluated on the hypothetical non-homogeneous aquifer of Rozos and Koutsoyiannis (2010) study. The size of this aquifer was 50×50 km^2, a constant uniform recharge was applied on it, a series of injecting wells with constant rate were deployed on the right half of its lower edge and a series of drains were deployed on the middle half of the upper edge.

Rozos and Koutsoyiannis (2010) employed four FVMSI meshes to simulate this aquifer. The FVMSI mesh that achieved the best performance in that study was used here. The mesh design was based on the equipotential lines (40 m interval, shown in the figure next) derived from the steady state solution of this aquifer with MODFLOW (grid of 100×100 cells). Flow lines were drawn manually perpendicular to equipotential lines. These two line sets define the FVMSI cells that discretize the flow domain.



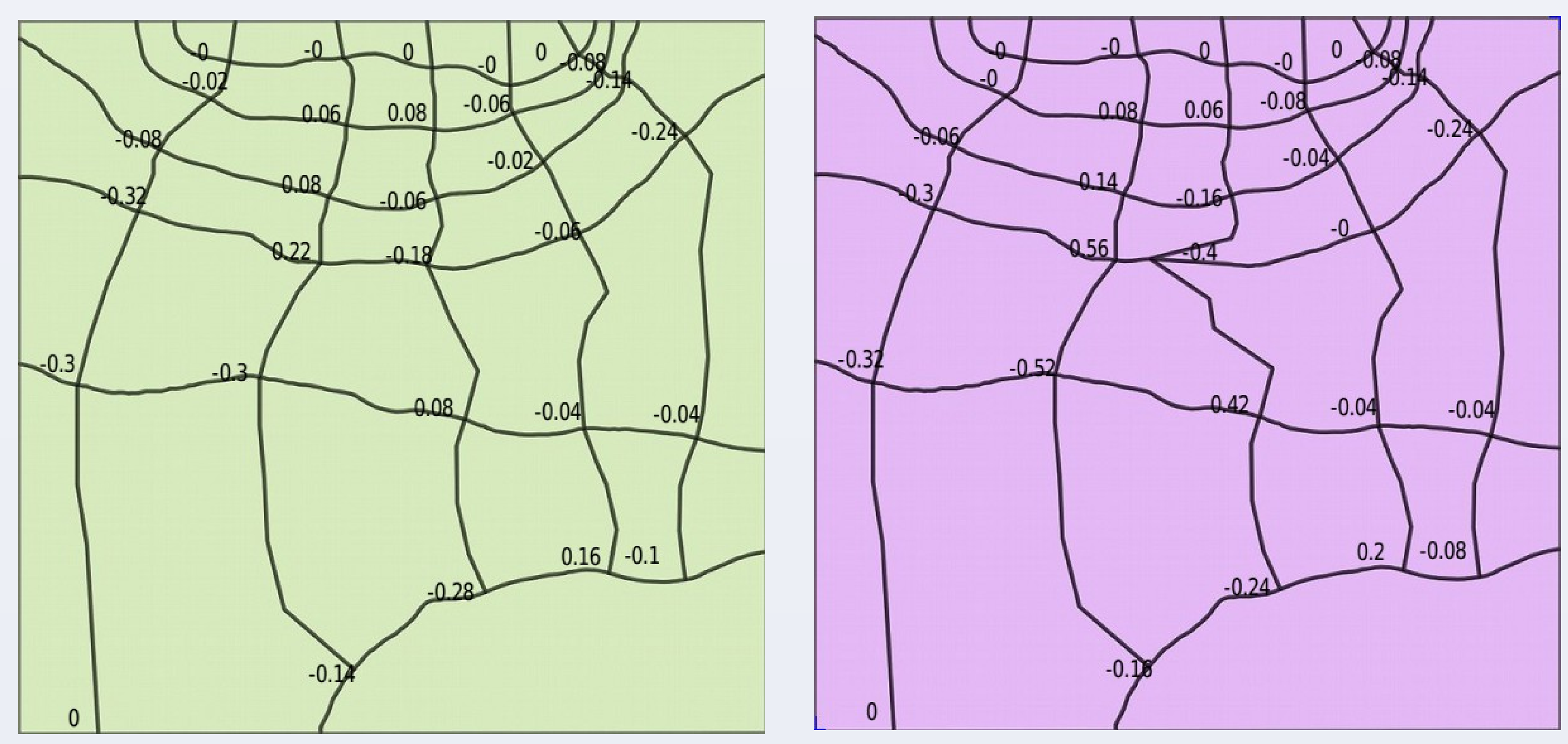
Equipotential lines used to draw the FVMSI mesh.

This design method promotes the compliance of a mesh to the two FVMSI conditions, which are: (i) cell boundaries either flow or equipotential lines, and (ii) the longitudinal fluxes should be parallel to the lines connecting the centres of the corresponding adjacent cells and perpendicular to their common edge.

Afterward, the mesh produced with this design method (called original hereafter) was deliberately deformed to introduce non conformity error.

Results

The original and the deformed meshes (shown left and right respectively in the figure below) were solved with the FVMSI model described in Rozos and Koutsoyiannis (2010) and the results of the solutions were compared against reference values obtained from the solution of the aquifer with MODFLOW. The RMSE of the solutions of these two meshes were found to be 5.32 and 6.46 m respectively. The difference between these RMSE values indicates the impact of the deliberately introduced non-conformity error.



Original (left) and deformed (right) meshes with ratios of lateral to longitudinal fluxes.

Using GNU Octave and QGIS, the ratio of all fluxes ρ was calculated. More specifically, the ratio was calculated with the formula:

$$\rho_{ij} = \frac{q_{ij,ij+1}/K_{ij,ij+1/2}}{q_{ij,i+1j}/K_{ij,i+1/2j}}$$

where q with index ($ij,ij+1$) refers to the flux between cell j of the row i and the next cell of the same row, q with index ($ij,i+1j$) refers to the flux between cell j of the row i and the cell j of row $i+1$, K with index ($ij,ij+1/2$) refers to the conductivity between cell j of the row i and the next cell of the same row, K with index ($ij,i+1/2j$) refers to the conductivity between cell j of the row i and the cell j of row $i+1$.

The calculated values of ρ for all cells are displayed in the two figures above. A close inspection of these values reveals that the absolute value of ρ is increased around the cells that were deliberately deformed. High absolute values can be found surrounding one more cell also (both in original and deformed mesh). This cell, of which the geometry was not affected by the deliberate deformation, is the second cell in the second complete row from the bottom (surrounded by ρ values -0.32, 0.22, -0.3 and -0.3 in the original mesh). These high values can be attributed to the fact that the lines that connect the centre of this cell with the centres of the cells above and below deviate significantly from the corresponding perpendicular lines between their common edges (2nd FVMSI condition). Another reason that may explain these high ρ values is that the length (size along the flow lines) of these three cells increases with an expansion factor greater than 2, whereas guidelines for efficient discretization suggest that this factor should be between 1.2 and 1.5 (Barrash and Dougherty, 1997).

Conclusions

- A close inspection of the definition of the ratio ρ reveals that it is equivalent to the ratio of the lateral to longitudinal gradients (the ratio of magnitudes). Since the lateral gradient (i.e. the gradient along the equipotential lines) is expected to be zero, the ratio ρ should also be zero if the mesh design is sound.
- Therefore, high ρ values are met in the parts of a mesh that need improvement.
- The proposed method is self-sufficient requiring no external information, but only the results of the simulation of the FVMSI mesh.
- This method deals only with errors due to non-conformity to FVMSI conditions. Other type of errors (e.g. truncation error, representation error, etc) do plague FVMSI (like FDM, FEM, etc) but cannot be detected by this method.

References

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