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A stochastic approach to causality

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Outline

1. Some background and probabilistic conceptions of causality
2. Motivating a stochastic approach to necessary conditions
3. The response function characterization of causality
4. How it works: artificial examples
5. Application

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Causality: contemporary approaches

Henry Mehlberg (1904-1979)

Causal Theory of Time: *No causal process (i.e., such that of two consecutive phases, one is always the cause of the other) can be reversible*

→ Irreversibility

Patrick Suppes (1922-2014)

An event $B_{t'}$ [occurring at time t'] *is a* prima facie *cause of the event* A_t [occurring at time t] *if and only if*

(i) $t' < t$,

(ii) $P(B_{t'}) > 0$,

(iii) $P(A_t|B_{t'}) > P(A_t)$.

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Brian Skyrms (1938-)

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Probabilistic Law

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David Cox (1924-2022)

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→ Probabilistic Law

(iv′) there is no event $C_{t''}$ at time $t'' < t' < t$ which "screens off" $B_{t'}$ from A_t such that $P(A_t|B_{t'}C_{t''}) = P(A_t|C_{t''})$.

→ Avoid spurious correlations

Causality: from probabilities to stochastics

◊ Probabilistic characterisations of causality are fine for reproducible events. This means that they are useful for:

- events that are controlled within the environment of a laboratory or
- events that are described sufficiently broadly that they actually re-occur (e.g. storm, flood,…)

◊ If a *more precise* quantification of events that occur in *open systems* is required, it will be the case that:

- several causal factors over which we have no control will be involved
- the events are not reproducible

◊ This suggests seeking necessary conditions:

- for one among many other possibly unknown causes;
- that apply to time-series of events.

A popular option: Granger causality

◊ **Clive Granger** (1934-2009) devised a statistical test for the claim that time-series $\{X_t\}$ has information useful to predict $\{Y_t\}$ or "Granger-causes" Y_t.

◊ The null hypothesis of no-Granger causality is:

$$b_p = b_{p+1} = \cdots = b_q = 0$$

where:

$$Y_t = a_0 + \sum_{i=1}^{m} a_i \, Y_{t-i} + \sum_{i=p}^{q} b_i \, X_{t-i} + W_t$$

This is tested with an F-test.

◊ It is questionable whether causality is best detected as what, *additionally* to a signal's correlation structure, improves forecasting. The method we propose below does not therefore include any autoregressive terms.

From first principles to a necessary condition (1)

◊ As starting point, we take the key requirements that causality (i) is *law-governed* and (ii) defines an *irreversible* temporal order. For quantities X and Y for which time-series of observations are available, the first causes the second only if:

$$\delta y(t) = f_h(\delta x(t-h))\Delta h$$

where $h \geq 0$ (*irreversibility*) and Δh represents the time during which the causal effect is brought about and f_h is some function that will define the *causal law* and for which, assuming a single cause: $f_h(0) = 0$

◊ By Taylor expansion:

$$\delta y(t) = \delta x(t-h)\frac{df_h}{dx}(0)\Delta h + o(\delta x(t-h))\Delta h$$

and if we define $g(h) = \frac{df_h}{dx}(0)$, we obtain:

$$\delta y(t) = \delta x(t-h)g(h)\Delta h + o(\delta x(t-h))\Delta h$$

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From first principles to a necessary condition (2)

◊ Representing the negligible terms as random terms $W(h)_t$, we get:

$$Y(t) = X(t-h)\,g(h)\Delta h + W(h)_t \Delta h$$

◊ Assuming now that X over a range of past times causes Y, by integration:

$$\boxed{Y(t) = \int_0^{\infty} X(t-h)\,g(h)dh + V(t)}$$

for some r.v. V(t). Function g is the *Impulse Response Function* (IRF).

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The task is to identify function g such that $Y(t) = \int_{-\infty}^{+\infty} X(t-h)\, g(h) dh + V(t)$,

The explained variance is $e = 1 - \frac{Var(V)}{Var(Y)}$

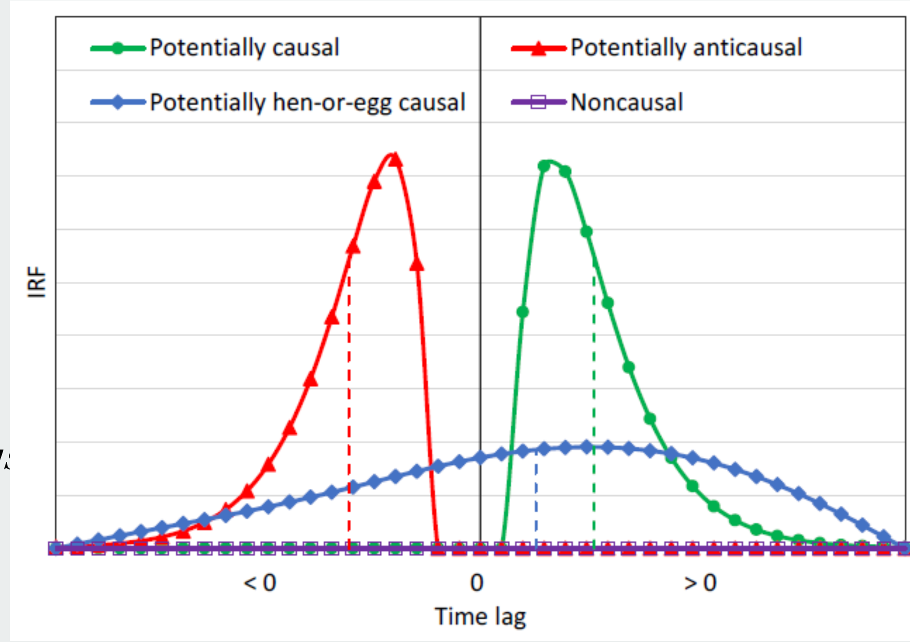
Necessary conditions:

◊ (X, Y) is *potentially causal* if $g(h)$=0 for any h<0 and e is non negligible;

◊ (X, Y) is *potentially anti-causal* if $g(h)$=0 for any h>0 and e is non-negligible ($\Longrightarrow$ (Y, X) is potentially causal);

◊ (X, Y) is *potentially hen-or-egg* (HOE) *cau* if $g(h)\neq$0 for some h>0 and some h<0, and e is non-negligible;

◊ (X, Y) is *non-causal* if e is negligible



There are infinitely many IRFs satisfying $Y(t) = \int_{-\infty}^{+\infty} X(t-h)\, g(h) dh + V(t)$ so additional requirements are added for the identification of g.

Additional requirements

◊ $g(h) \geq 0$ for all $h \in \mathcal{H}$

◊ The smoothness of the IRF, defined as $\mathrm{E} = \int_{-\infty}^{+\infty} (g''(h))^2\, dh$ must be smaller than some pre-defined value E_0

◊ $\mathrm{Var}(\mathrm{V})$ must be minimal

$$Y(t) = \int_{-\infty}^{+\infty} X(t-h)\, g(h) dh + V(t)$$

is then discretised as:

$$Y_t = \sum_{-\infty}^{+\infty} X_{t-j}\, g_j + V_t$$

This is estimated through:

$$\widehat{y_t} = \sum_{-J}^{+J} x_{t-j}\, g_j + \mu_v$$

where μ_v ensures that the estimation is unbiased. The IRF is then estimated from:

$$\boxed{\begin{array}{c}\text{Min}\,\{\text{var}(\widehat{y_t} - y_t)\} \\ \text{s.t. } E \le E_0;\ (\forall j)\ g_j \ge 0\end{array}}$$

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Artificial Examples

We construct artificial systems by using the equation:

$$Y_t = \sum_{i=0}^{+I_H} a_i X_{t-i} + U_t, \text{ with } U_t \sim N(0, 0.5^2)$$

where I_H varies according to the application and X_t is defined as:

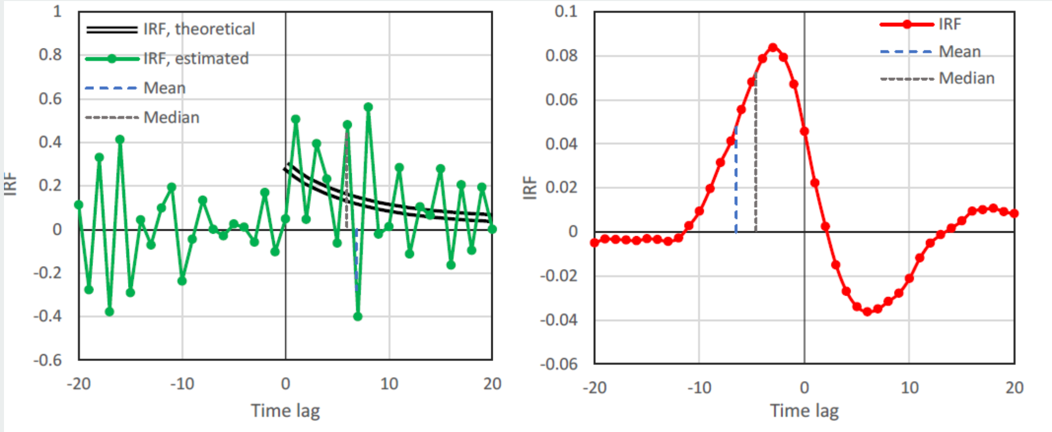
$$X_t = \sum_{i=-I}^{+I} a_i w_{t-i} \text{ where } w_t \sim N(0,1), I = 1024; (\forall i)\ a_i = a_{-i} \text{ from an FHK-C}$$

Causal system #1

$\{I_H = 20$; no constraints; $J = 20\}$

Left: $x \rightarrow y$ $(e = 0.94)$

Right: $y \rightarrow x$ $(e = 0.97)$



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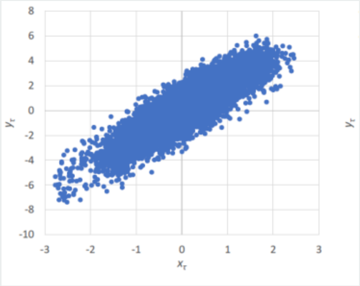
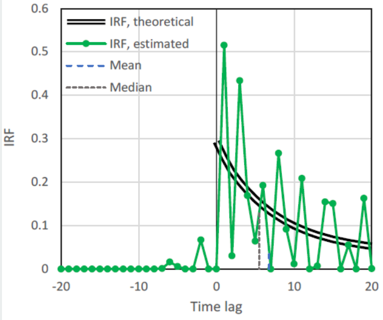


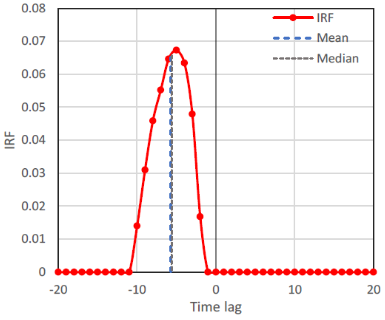
Causal system #2

$\{I_H = 20$; non-negativity; no roughness constraint; $J = 20\}$

Left: $x \rightarrow y$ ($e = 0.94$)

Right: y$\rightarrow x$ ($e = 0.94$)



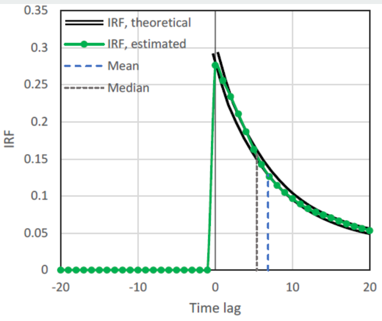


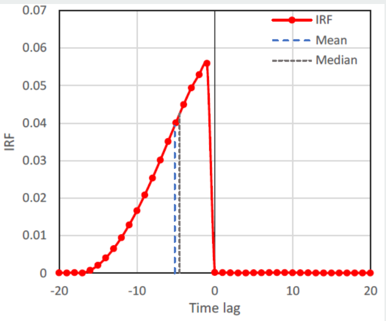
Causal system #3

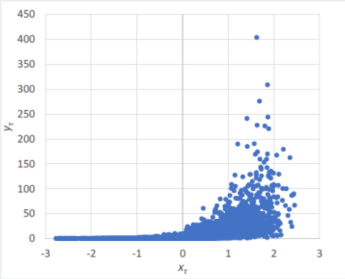
$\{I_H = 20$; non-negativity; roughness constraint ; $J = 20\}$

Left: $x \rightarrow y$ ($e = 0.94$)

Right: y$\rightarrow x$ ($e = 0.94$)





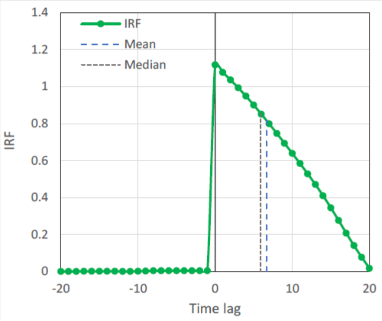


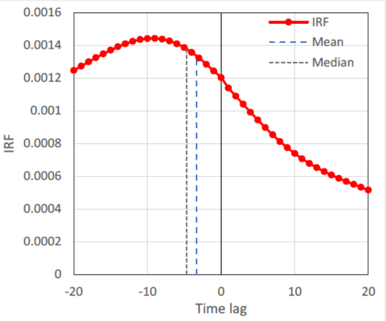
Causal system #4

$\{I_H = 20$; non-negativity; roughness constraint; ; $J = 20$; $e^y\}$

Left: $x \rightarrow y$ ($e = 0.32$)

Right: y$\rightarrow x$ ($e = 0.43$)





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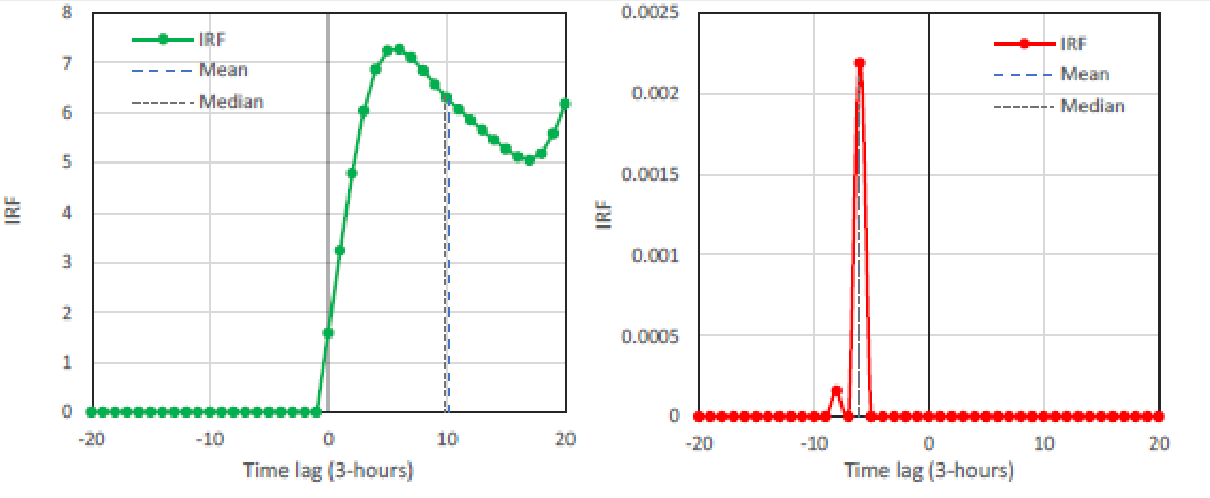
Application

Precipitation and Runoff

{non-negativity; roughness constraint; $J = 20$}

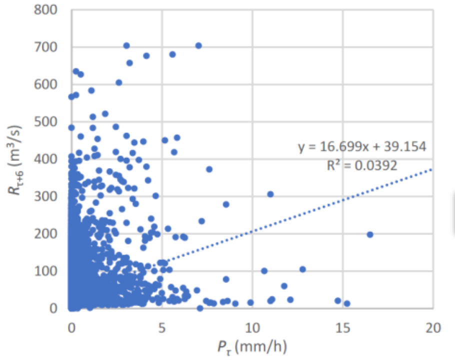
Left: $P \rightarrow R$ ($e = 0.17$)

Right: $R \rightarrow P$ ($e = 0.04$)

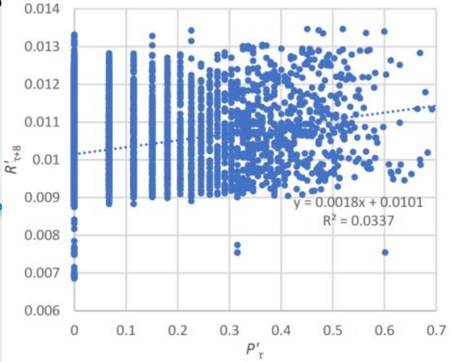


Time-step: 3 hours

Problem: Non-linearity



$$P' = c_P \ln\left(1 + \frac{P}{c_P}\right), \qquad R' = c_R \ln\left(1 + \frac{R}{c_R}\right)$$



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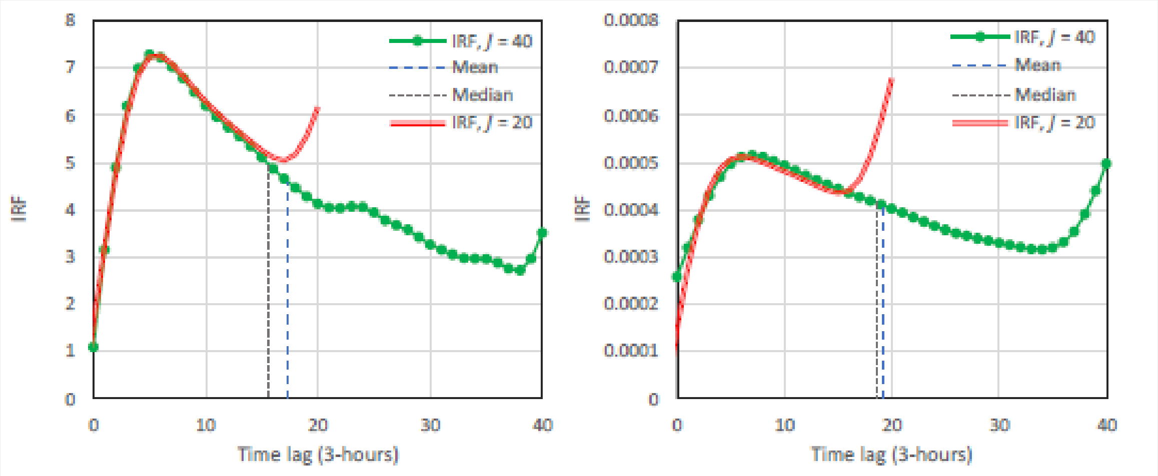
Precipitation and runoff
(continued)
{non-negativity; roughness constraint; $J = 20;\ 40$}
Left: $x \rightarrow y$ untransformed ($e = 0.17;\ 0.26$)
Right: $x \rightarrow y$ transformed ($e = 0.68; 0.71$)



Note the effect of taking a longer window (± 40 instead of ± 20) for the definition of the IRF.

Concluding remarks

- We have proposed conditions that need to be fulfilled to claim that there is causality in non-oscillatory open systems.
- These are necessary but not sufficient and there is a degree of subjectivity in the conclusions since no statistical test has been developed
- More information and examples are found in our papers

References

Koutsoyiannis, D., Onof, C., Christofides, A., & Kundzewicz, Z. W. (2022). Revisiting causality using stochastics: 1. Theory. *Proceedings of the Royal Society A*, *478*(2261), 20210835

Koutsoyiannis, D., Onof, C., Christofides, A., & Kundzewicz, Z. W. (2022). Revisiting causality using stochastics: 2. Applications. *Proceedings of the Royal Society A*, *478*(2261), 20210836.