A stochastic approach to causality

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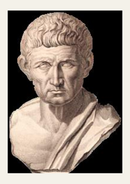
Background

Causality: a philosophical puzzle



Aristotle (384-322 BC)

That which when present is the cause of something, when absent we <u>*sometimes*</u> *consider to be the cause of the contrary.* ➡ Probabilities

David Hume (1711-1776)

Custom alone makes us <u>*expect*</u> *for the future, a similar train of events with those which have appeared in the past.* ➡ Only subjective?

Immanuel Kant (1724-1804)

All alterations occur in accordance with the <u>*law*</u> *of the connection of cause and effect.* ➡ Objective

It is really this <u>*necessitation*</u> *that first makes possible the representation of a* <u>*succession*</u>*.* ➡ Irreversibility

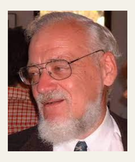
Causality: contemporary approaches

Patrick Suppes (1922-2014)

An event $B_{t'}$ [occurring at time t'] *is a* prima facie *cause of the event* A_t [occurring at time t] *if and only if*

(i) $t' < t$,
(ii) $P(B_{t'}) > 0$,
(iii) $P(A_t|B_{t'}) > P(A_t)$.

➡ Probabilistic law



Brian Skyrms (1938-)

Alternative third condition:

$P(A_t|B_{t'}) > P(A_t|\overline{B}_{t'})$

➡ Conditional probabilities

David Cox (1924-2022)

(iv′) there is no event $C_{t''}$ at time $t'' < t' < t$ which "screens off" $B_{t'}$ from A_t such that $P(A_t|B_{t'}C_{t''}) = P(A_t|C_{t''})$.

➡ Avoid spurious correlations

We seek *necessary* conditions of causality accounting for its being *law-governed* and *irreversible*, These conditions must define the *conditional* dependence of effect upon cause in probabilistic terms, while excluding *spurious correlations* as far as possible.

However:

Descriptions in terms of probabilities of events are fine for events defined sufficiently broadly (e.g. flood/no flood) and for reproducible events that are controlled in the lab.

For more precise quantifications in open systems, it is better to seek *causal links between time-series*.

Clive Granger (1944-2009)

Time-series $\{X_t\}$ has information useful to predict $\{Y_t\}$ or "Granger-causes" Y_t.

The null hypothesis of no-Granger causality is:

$$b_p = b_{p+1} = \cdots = b_q = 0$$

where:

$$Y_t = a_0 + \sum_{i=1}^{m} a_i\, Y_{t-i} + \sum_{i=p}^{q} b_i\, X_{t-i} + W_t$$

This is tested with an F-test.

But causality is not best defined as what, *additionally* to a signal's correlation structure, improves forecasting.

Alternative time-series proposal

Motivation

◊ As starting point, we take the key requirements that causality (i) is *law-governed* and (ii) defines an *irreversible* temporal order. For quantities X and Y for which time-series of observations are available, the first causes the second only if:

$$\delta y(t) = f_h(\delta x(t-h))\Delta h$$

where $h \geq 0$ (*irreversibility*) and Δh represents the time during which the causal effect is brought about and f_h is some function that will define the *causal law* and for which, assuming a single cause: $f_h(0) = 0$

◊ By Taylor expansion:

$$\delta y(t) = \delta x(t-h)\frac{df_h}{dx}(0)\Delta h + o(\delta x(t-h))\Delta h$$

and if we define $g(h) = \frac{df_h}{dx}(0)$, we obtain:

$$\delta y(t) = \delta x(t-h) g(h)\Delta h + o(\delta x(t-h))\Delta h$$

◊ Representing the negligible terms as random terms $W(h)_t$, we get: $Y(t) = X(t-h)\, g(h)\Delta h + W(h)_t \Delta h$

◊ Assuming now that X over a range of past times causes Y, by integration:

$$Y(t) = \int_0^{\infty} X(t-h)\, g(h) dh + V(t)$$

Function g is the *Impulse Response Function* (IRF).

Necessary conditions

The task is to <u>identify function</u> g such that

$$Y(t) = \int_{-\infty}^{+\infty} X(t-h)\, g(h) dh + V(t)$$

The explained variance is $e = 1 - \frac{Var(V)}{Var(Y)}$

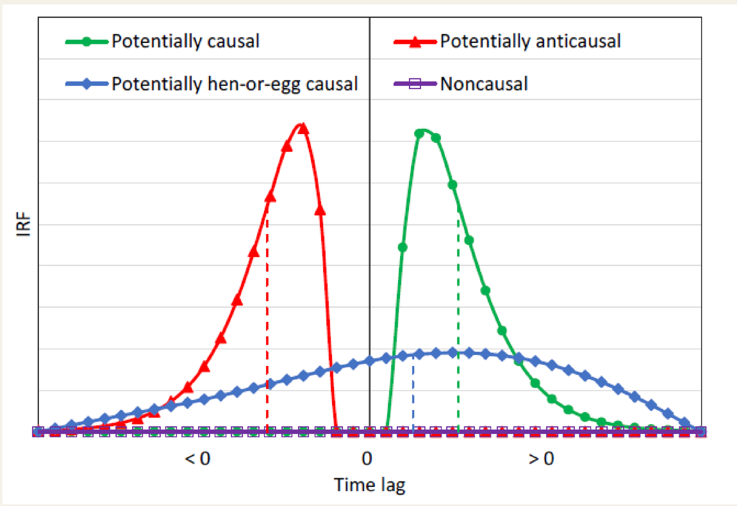
◊ (X, Y) is ***potentially causal*** if $g(h)$=0 for any h<0 and e is non negligible;

◊ (X, Y) is ***potentially anti-causal*** if $g(h)$=0 for any h>0 and e is non-negligible ($\Longrightarrow$ (Y, X) is potentially causal*)*;

◊ (X, Y) is ***potentially hen-or-egg*** (*HOE*) *causal* if $g(h)$≠0 for some h>0 and some h<0, and e is non-negligible;

◊ (X, Y) is ***non-causal*** if e is negligible

Plot of the IRF g as a function of lag h



Additional requirements for potential causality

◊ $g(h) \geq 0$ for all $h \in \mathcal{H}$

◊ The smoothness of the IRF, defined as $\mathrm{E} = \int_{-\infty}^{+\infty} (g''(h))^2\, dh$ must be smaller than some pre-defined value E_0

◊ Var(V) must be minimal

Estimation

$$Y(t) = \int_{-\infty}^{+\infty} X(t-h)\, g(h) dh + V(t)$$

is discretised as:

$$Y_t = \sum_{-\infty}^{+\infty} X_{t-j}\, g_j + V_t$$

This is estimated through the following estimator:

$$\hat{y}_t = \sum_{-J}^{+J} x_{t-j}\, g_j + \mu_v$$

where μ_v ensures that the estimation is unbiased.

◊ The IRF is estimated by minimizing the sample variance of $\hat{y}_t - y_t$ while keeping the roughness index smaller than E_0.

This also yields: $\hat{e} = 1 - \frac{var(\hat{y}_t - y_t)}{var(y_t)}$

Artificial examples (1)

Construction

We construct artificial systems by using the equation:

$$Y_t = \sum_{i=-I_L}^{+I_H} a_i X_{t-i} + U_t$$

with $U_t \sim N(0, 0.5^2)$, where I_L and I_H vary according to the application and X_t is a Filtered Hurst Kolmogorov process.

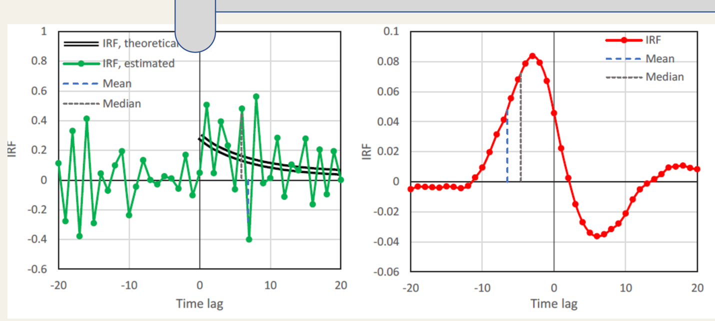
Systems #1 - #3 show the role played by the above additional requirements in identifying the IRF

Causal system #1

$\{I_L = 0; I_H = 20$; no constraints; $J = 20\}$

Left: $x \to y$ $(e = 0.94)$

Right: y→ x $(e = 0.97)$

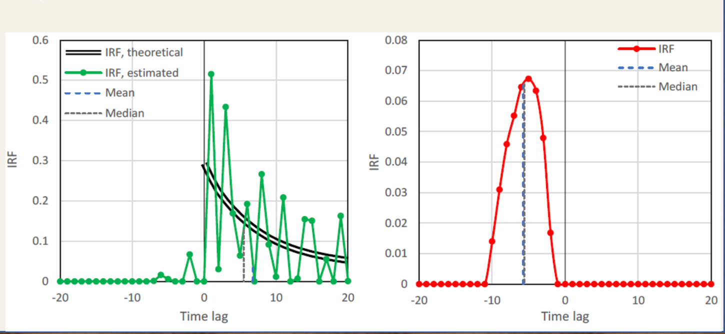


Causal system #2

$\{I_L = 0; I_H = 20$; non-negativity; no roughness constraint; $J = 20\}$

Left: $x \to y$ $(e = 0.94)$

Right: y→ x $(e = 0.94)$



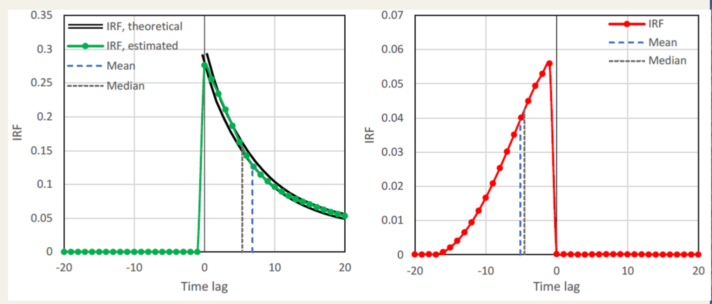
Artificial examples (2)

Causal system #3

$\{I_L = 0; I_H = 20$; non-negativity; roughness constraint ; $J = 20\}$

Left: $x \to y$ $(e = 0.94)$

Right: y→ x $(e = 0.94)$

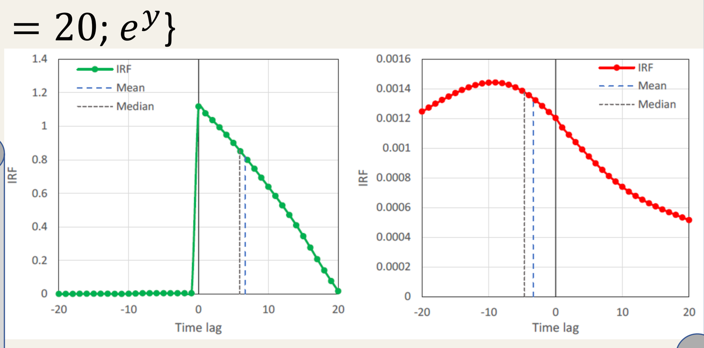


Causal system #4

$\{I_L = 0; I_H = 20$; non-negativity; roughness constraint; ; $J = 20$; $e^y\}$

Left: $x \to y$ $(e = 0.32)$

Right: $y \to x$ $(e = 0.43)$



In system #4, y is exponentiated. Although e is not large, causality is detected.

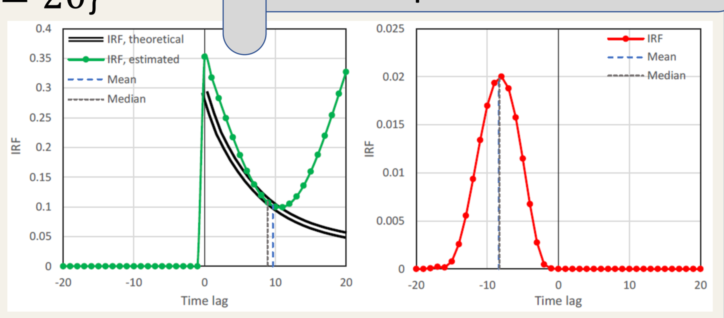
Causal system #5

$\{I_L = 0; I_H = 1024$; non-negativity; roughness constraint; $J = 20\}$

Left: $x \to y$ $(e = 0.57)$

Right: y→ x $(e = 0.50)$

In system #5, the$\pm$20 window is too small to capture the full causal effect which spans 1024 time steps.



Real examples

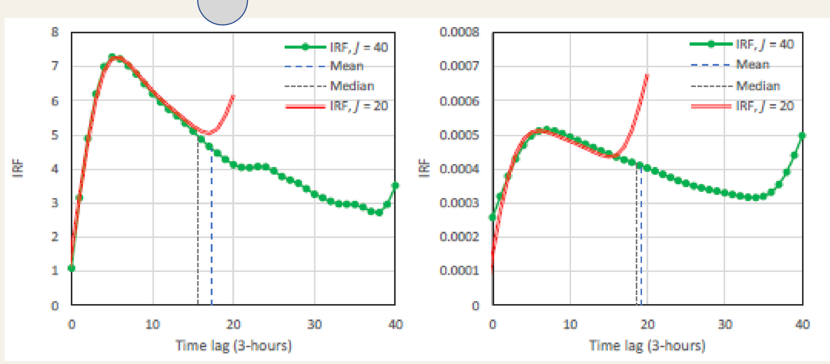
Precipitation and runoff

{non-negativity; roughness constraint; $J = 20;\ 40\}$

Left: $x \to y$ untransformed $(e = 0.17;\ 0.26)$

Right: $x \to y$ transformed $(e = 0.68; 0.71)$

x and y are 3-hr precipitation and runoff. Because they are non-linearly related, a nonlinear transform raises e. Note also the impact of window size $(\pm 20, \pm 40)$. Clear potential causality.



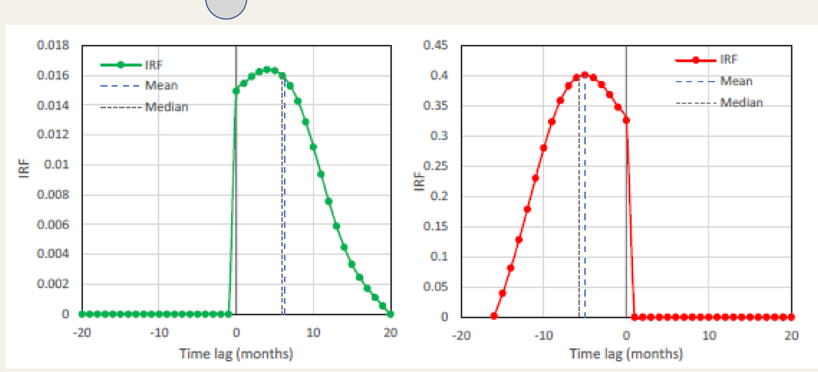
Atmospheric Temperature and ENSO

{non-negativity; roughness constraint; $J = 20\}$

Left: ENSO $\to T$ $(e = 0.39)$

Right: $T \to$ ENSO $(e = 0.30)$

x and y are monthly ENSO and atmospheric temperature (left) and vice-versa (right). Again, there is clear evidence of potential causality ENSO → T



Conclusions

- We have proposed conditions that need to be fulfilled to claim that there is causality in non-oscillatory open systems.
- These are necessary but not sufficient and there is a degree of subjectivity in the conclusions since no statistical test has been developed
- More information and examples are found in our papers

References

Koutsoyiannis, D., Onof, C., Christofides, A., & Kundzewicz, Z. W. (2022). Revisiting causality using stochastics: 1. Theory. *Proceedings of the Royal Society A*, *478*(2261), 20210835

Koutsoyiannis, D., Onof, C., Christofides, A., & Kundzewicz, Z. W. (2022). Revisiting causality using stochastics: 2. Applications. *Proceedings of the Royal Society A*, *478*(2261), 20210836.