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On the Parametric Approach to Unit Hydrograph Identification

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Abstract. Unit hydrograph identification by the parametric approach is based on the assumption of a proper analytical form for its shape, using a limited number of parameters. This paper presents various suitable analytical forms for the instantaneous unit hydrograph, originated from known probability density functions or transformations of them. Analytical expressions for the moments of area of these form versus their definition parameters are theoretically derived. The relation between moments and specific shape characteristics are also examined. Two different methods of parameter estimation are studied, the first being the well-known method of moments, while the second is based on the minimization of the integral error between derived and recorded flood hydrographs. The above tasks are illustrated with application examples originated from case studies of catchments in Greece.

Key words. Unit hydrograph, instantaneous unit hydrograph, identification, probability density function, probability distribution function, method of moments, optimization.

0. Notations

A	catchment area
a, b, c	definition parameters (generally a is a scale parameter, while b and c are shape parameters)
C_v	coefficient of variation
C_s	skewness coefficient
D	net rainfall duration
$f(\ )$	probability density function (PDF)
$F(\ )$	cumulative (probability) distribution function (CDF)
$g(\ )$	objective function
H	net rainfall depth
H_0	unit (net) rainfall depth (= 10 mm)
$I(t)$	net hyetograph
$i(t)$	standardized net hyetograph (SNH)
I_n	n^{th} central moment of the standardized net hyetograph
$Q(t)$	surface runoff hydrograph
$q(t)$	standardized surface runoff hyrograph (SSRH)
Q_n	n^{th} central moment of the standardized surface runoff hydrograph
$S_D(t)$	S-curve derived from a unit hydrograph of duration D

$s(t)$	standardized S-curve (SSC)
t	time
T_D	flood duration of the unit hydrograph $U_D(t)$
T_0	flood duration of the instantaneous unit hydrograph $U_0(t)$ (= right bound of the function $U_0(t)$)
t_U	IUH lag time (defined as the time from the origin to the center of area of IUH or SIUH)
t_I	time from the origin to the center of the area of the net hyetograph
t_Q	time from the origin to the center of the area of the surface runoff hydrograph
t_p	time from the origin to the peak of IUH (or SIUH)
$U_D(t)$	unit hydrograph for rainfall of duration D (DUH)
$U_o(t)$	instantaneous unit hydrograph (IUH)
$u(t)$	standardized instantaneous unit hydrograph (SIUH)
U_n	nth central moment of area of IUH
U'_n	nth moment of IUH about the origin
U''_n	nth moment of IUH about the right bound (when exists)
V	surface runoff volume
V_0	volume corresponding to the unit hydrograph

1. Introduction

Common mathematical approaches to model synthesis, include (1) discretization techniques, i.e. determination of the model in a finite number of discrete points, and (2) parametric techniques, i.e. the assumption of a proper analytical form for the model, with a limited number of parameters, and the estimation of these parameters by means of known restrictions and/or the optimization of an objective function.

Both the above approaches have been used for the unit hydrograph (UH) synthesis. The first has become the most common method and is based on linear analysis (matrix inversion technique – O'Donnell (1986)). The second was founded by Nash (1959), who showed that the moments of the area of the instantaneous unit hydrograph (IUH) can be easily derived from recorded hydrographs and simultaneous rainfall records. Nash also studied several suitable two-parameter analytical forms to represent the shape of the IUH, the most common being the gamma-PDF form. The parametric approach has also been used in synthetic unit hydrograph derivations, with the most common analytical forms being the triangular and the gamma-PDF.

The first approach is generally considered as more accurate, because of the considerable number of points defining the UH, while the second uses a very limited number of parameters (2 or 3) for UH shape representation. In fact, in the parametric approach the inaccuracies due to the limited number of parameters are minor when compared with the uncertainties of the whole process of UH derivation from recorded data, which are met in the areal rainfall estimation, the establishment of the stagedischarge relationship, baseflow separation, and, finally, separation and time distribution of rainfall losses.

Moreover, it is often very difficult or almost impossible to find recorded flood events which originated from entirely uniform over the catchment rainfall. Finally, the assumption of the catchment linearity is not strictly valid and the catchment response is not unique. Because of the above uncertainties, it is a very common practice in the formation of design flood hydrographs, to consider a unit hydrograph more severe than that derived from flow records (e.g. by reducing the time to peak by 2/3, see Sutcliffe (1978)).

The above discussion indicates that the parametric method, although seemingly less accurate than the standard linear method, can be a good approximation to the unit hydrograph identification of real-world catchments, since the inaccuracies introduced by the use of a limited number of parameters are minor. Moreover, the parametric method has some advantages which will be discussed later.

Problems associated with the application of the parametric approach are the selection of the proper analytical form for the UH representation and, mainly, the method for parameter estimation. These problems are systematically examined throughout this study.

2. Definitions and General Relations

Let $U_D(t)$ be the unit hydrograph for a net rainfall of duration D (DUH). The instantaneous unit hydrograph $U_0(t)$ corresponds to the case where $D = 0$. We denote by V_0 the surface runoff volume corresponding to the unit rainfall with depth $H_0 = 10$ mm, that is

$$V_0 = \int_0^{T_D} U_D(t)\,\mathrm{d}t = \int_0^{T_0} U_0(t)\,\mathrm{d}t = H_0 \cdot A,$$

where T_D and T_0 are large enough time intervals, referred to as flood durations (theoretically, the right bounds for the functions $U_D(t)$ and $U_0(t)$, respectively, which can be equal to ∞), and A the catchment area. Now we define the function

$$u(t) = U_0(t) \,/\, V_0 \tag{1}$$

which will be referred to as *standardized instantaneous unit hydrograph* (SIUH). This is a positive function, with the dimension $(\text{time})^{-1}$, which has the property

$$\int_0^{T_0} u(t)\,\mathrm{d}t = 1 \cdot \tag{2}$$

In general, $u(t)$ is a single peaked function. The time to peak, t_p, and the peak value, $u_p = u(t_p)$, are the main characteristics of SIUH.

Furthermore, let $S_D(t)$ be the S-curve (DSC) derived from the DUH, which corresponds to a rainfall intensity equal to H_0/D, and of infinite duration. The DSC is related to DUH by

$$U_D(t) = S_D(t) - S_D(t\text{-}D) \tag{3}$$

and to IUH by

$$S_D(t) = \frac{1}{D} \int_0^t U_0(t)\,\mathrm{d}t. \tag{4}$$

Additionally, we define the function

$$s(t) = S_D(t) \cdot (D/V_0) \tag{5}$$

which is dimensionless, independent of the duration D, and has the properties

$$s(0) = 0\,, \qquad s(T_0) = 1. \tag{6}$$

This function will be referred to as the *standardized S-curve* (SSC). SSC and SIUH are related by

$$s(t) = \int_0^t u(t)\,\mathrm{d}t, \tag{7}$$

$$u(t) = \frac{\mathrm{d}s(t)}{\mathrm{d}t}\,. \tag{8}$$

Finally, let U_n be the nth central moment of area of the function $u(t)$, U'_n the nth moment about the origin, and U''_n the nth moment about the right bound T_0 (if it exists). These are defined by

$$U_n = \int_0^{T_0} (t - t_U)^n\, u(t)\,\mathrm{d}t, \tag{9}$$

$$U'_n = \int_0^{T_0} t^n\, u(t)\,\mathrm{d}t, \tag{10}$$

$$U''_n = \int_0^{T_0} (T_0 - t)^n\, u(t)\,\mathrm{d}t. \tag{11}$$

where $t_U = U'_1$ is the distance of the center of the area of SIUH from the origin, known as *lag time*. (Note the term (T_0-t) in (11), which is opposite to the usual.)

Each family of moments can theoretically determine the complete shape of SIUH, when an infinite number of them is known. In reality, only a limited number can be estimated, but nevertheless, this limited number holds substantial information about the shape, which is very helpful for IUH identification.

Given a specific analytical form of SIUH, it is an easy matter to derive the DUH for any duration D. This can be done by a subsequent application of Equations (7), (5) and (3).

3. Analytical Forms for the SIUH Representation

The functions $u(t)$ and $s(t)$ are mathematically similar to the families of probability density functions (PDFs) and distribution functions (CDFs). Those single-peaked PDFs, which are left-bounded by zero, are ideal for the representation of the SIUH.

Normally they should have a right bound, too, but this is not necessarily considered as a strict theoretical requirement, since all PDFs tend to zero for large values.

Eight particular analytical forms have been systematically examined in this study - all of them originating from know probability density functions (PDFs) of their transformations. They have two or three parameters and are left-bounded by zero or double-bounded. These forms are described below. In their analytical expressions, a generally denotes a scale parameter, while b and c denote shape parameters. The expressions of their main features (theoretically derived in the present study, except those of well-known functions) are summarized in Tables I and II.

1. *Double Triangular (DT)* (double-bounded/two-parameter)

This form is, in fact, a single triangle consisting of two successive triangular PDFs (thus, the characterization 'double'), the first with a negative skewness and the second with a positive one. The SIUH is expressed by

$$u(t) = \begin{cases} \dfrac{2t/a}{ab}, & 0 \leqslant t/a \leqslant b, \\ \\ \dfrac{2(1-t/a)}{a(1-b)}, & b \leqslant t/a \leqslant 1. \end{cases} \tag{12}$$

The double-triangular form has been widely used for the expression of synthetic unit hydrographs (for example, see Sutcliffe (1978)), but not so much for the IUH itself.

2. *Gamma* (Γ) (left-bounded/two-parameter)

This form has been suggested by Nash (1959), and is the most common for the IUH analytical expression, either synthetic or from recorded data. Its analytical expression is

$$u(t) = \frac{(t/a)^{b-1}\ e^{-t/a}}{a\Gamma(b)}, \quad t \geqslant 0. \tag{13}$$

3. *Log-Normal (LN)* (left-bounded/two-parameter)

This form was also suggested by Nash (1959); its analytical expression is

$$u(t) = \frac{1}{t(\pi b)^{1/2}} \exp\left[\frac{-\ln^2(t/a)}{\mathrm{b}}\right], \quad t \geqslant 0. \tag{14}$$

4. *Weibull (W)* (left-bounded/two-parameter)

Originating from the Weibull distribution function, we get the following form for SSC

$$s(t) = 1 - e^{-(t/a)^b}, \quad t \geqslant 0. \tag{15}$$

Table I. Two parameter IUH analytical forms

Form →	Double-Triangular (DT)	Gamma (Γ)	Log-normal (LN)	Weibull (W)
Standardized IUH and S-curve	$u(t) = \begin{cases} \frac{2t/a}{ab}, & 0 \leqslant t \leqslant ba \\ \frac{2(1-t/a)}{a(1-b)}, & ba \leqslant t \leqslant a \end{cases}$ $s(t) = \begin{cases} \frac{(t/a)^2}{b}, & 0 \leqslant t \leqslant ba \\ 1 - \frac{(1-t/a)^2}{1-b}, & ba \leqslant t \leqslant a \end{cases}$	$u(t) = \frac{(t/a)^{b-1}\ e^{-t/a}}{a\Gamma(b)}$ $s(t) = \frac{\gamma(t/a,b)}{\Gamma(b)}$	$u(t) = \frac{1}{t\sqrt{\pi b}} \exp\left[-\frac{\ln^2(t/a)}{b}\right]$ $s(t) = G\left[\frac{\ln(t/a)}{\sqrt{b/2}}\right]$	$u(t) = (b/a)(t/a)^{b-1}\ e^{-(t/a)^b}$ $s(t) = 1 - e^{-(t/a)^b}$
t-range parameters	$0 \leqslant t \leqslant a$ $a > 0$: scale parameter $b < 1$: shape parameter	$t \geqslant 0$ $a > 0$: scale parameter $b > 1$: shape parameter	$t \geqslant 0$ $a > 0$: scale parameter $b > 0$: shape parameter	$t \geqslant 0$ $a > 0$: scale parameter $b > 1$: shape parameter
Lag time t_U Moments of area U_n and/or U'_n and/or U''_n	$t_U = \frac{a}{3}(1+b)$ $U_2 = \frac{a^2}{18}(1-b+b^2)$ $U_3 = \frac{a^3}{135}(1+b)(1-b/2)(1-2b)$ In general $U''_n = \frac{2a^n}{(n+1)(n+2)} \frac{1-b^{n+1}}{1-\mathrm{b}}$	$t_U = ab$ $U_2 = a^2 b$ $U_3 = 2a^3 b$ In general $U'_n = a^n b(b+1)\ldots(b+n-1)$	$t_U = a\ e^{b/4}$ $U_2 = a^2\ e^{b/2}\ (e^{b/2}-1)$ $U_3 = a^3\ e^{3b/4}\ (e^{b/2}-1)^2\ (e^{b/2}+2)$ In general $U'_n = a^n\ e^{n^2 b/4}$	$t_U = U_1 = a\Gamma(1+1/b)$ $U_2 = U'_2 - U'^2_1$ $U_3 = U'_3 - 3U'_2U'_1 + 2U'^3_1$ where $U'_2 = a^2\Gamma(1+2/b)$ $U'_3 = a^3\Gamma(1+3/b)$ In general $U'_n = a^n\Gamma(1+n/b)$

Variation C_v	$C_v = \sqrt{\frac{1-b+b^2}{2}} \frac{1}{1+b}$	$C_v = 1 / \sqrt{b}$	$C_v = \sqrt{e^{b/2}-1}$	$C_v = \sqrt{\frac{\Gamma(1+2/b)}{\Gamma^2(1+1/b)} - 1}$
Skewness C_s	$C_s = \frac{2\sqrt{2}}{5} \frac{(1+b)(1-b/2)(1-2b)}{(1-b+b^2)^{3/2}}$	$C_s = 2 / \sqrt{b} = 2C_v$	$C_s = \sqrt{e^{b/2}-1}\,[e^{b/2}+2]$ $= C_v^3 + 3C_v$	$C_s = U_3 / U_2^{3/2}$
Parameter estimation by the method of moments	$b = \frac{3 - \sqrt{24C_v^2 - 3}}{3 + \sqrt{24C_v^2 - 3}}$ $a = \frac{3t_U}{1+b}$ restriction: $1/\sqrt{8} \leqslant C_v \leqslant 1/\sqrt{2}$	$a = U_2 / t_U$ $b = t_U^2 / U_2$	$b = 2\ln(1+C_v^2)$ $a = t_U\, e^{-b/4} = \frac{t_U}{\sqrt{(1+C_v^2)}}$	b: by numerical solution of the equation: $\frac{\Gamma(1+2/b)}{\Gamma^2(1+1/b)} = C_v^2 + 1$ $a = \frac{t_U}{\Gamma(1+1/b)}$
Time to peak (mode)	$t_p = ab \quad U_p = 2/a$	$t_p = a(b-1)$	$t_p = a\, e^{-b/2}$	$t_p = a(1 - 1/b)^{1/b}$
Remarks		Definition of Γ-function $\Gamma(b) = \int_0^\infty x^{b-1} e^{-x}\, dx$ Definition of the incomplete γ-function $\gamma(y,b) = \int_0^y x^{b-1} e^{-x}\, dx$	Definition of the normal probability distribution function G $G(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2}\, dx$	

Note: The above given range for the parameters corresponds to the case that $u(t)$ is bell-shaped.

Table II. Three-parameter IUH analytical forms

Form →	Beta (B)	Double-power (DP)	Shifted Log-Pearson III (SLP)	Minus Log-Pearson III (MLP)
Standardized IUH and *S*-curve	$u(t) = \frac{(t/a)^{b-1}(1-t/a)^{c-1}}{a\,B(b,c)}$ $s(t) = \frac{\beta(t/a,b,c)}{B(b,c)}$	$u(t) = (bc/a)\,(1-t/a)^{b-1} \cdot$ $\cdot\,[1-(1-t/a)^b]^{c-1}$ $s(t) = [1-(1-t/a)^b]^c$	$u(t) = \frac{c^b}{a\Gamma(b)}\,\frac{[\ln(t/a+1)]^{b-1}}{(t/a+1)^{c+1}}$ $s(t) = \gamma[c \cdot \ln(t/a+1), b]/\Gamma(b)$	$u(t) = \frac{c^b (t/a)^{c-1}[-\ln(t/a)]^{b-1}}{a\Gamma(b)}$ $s(t) = \gamma[-c \cdot \ln(t/a), b]/\Gamma(b)$
t-range parameters	$0 \leqslant t \leqslant a$ $a > 0$: scale parameter $b,c > 1$: shape parameters	$0 \leqslant t \leqslant a$ $a > 0$: scale parameter $b,c > 1$: shape parameters	$t \geqslant 0$ $a > 0$: scale parameter $b > 1,\ c > 0$: shape parameters	$0 \leqslant t \leqslant a$ $a > 0$: scale parameter $b,c > 1$: shape parameters
Lag time t_U Moments of area U_n and/or U'_n and/or U''_n	$t_U = \frac{ab}{b+c}$ $U_2 = \frac{a^2bc}{(b+c)^2(b+c+1)}$ $U_3 = \frac{2a^3bc(c-b)}{(b+c)^3(b+c+1)(b+c+2)}$ In general $U'_n = \frac{a^n b(b+1)\,.\,.\,(b+n-1)}{(b+c)(b+c+1)\,.\,.\,(b+c+n-1)}$	$t_U = a - U''_1$ $U_2 = U''_2 - U''^2_1$ $U_3 = 3U''_2\,U''_1 - 2U''^3_1 - U''_3$ where $U''_1 = a^2c\,B(1+3/b,\ c)$ $U''_2 = a^2c\,B(1+3/b,\ c)$ $U''_3 = a^3c\,B(1+3/b,\ c)$ In general $U''_n = a^n c\,B[(1+\frac{n}{b},\ c]$	$t_U = U^*_1 - a$ $U_2 = U^*_2 - U^{*2}_1$ $U_3 = U^*_3 - 3U^*_2\,U^*_1 + 2U^{*3}_1$ where $U^*_1 = a\left[\frac{c}{c-1}\right]^b$ $U^*_2 = a^2\left[\frac{c}{c-2}\right]^b$ $U^*_3 = a^3\left[\frac{c}{c-3}\right]^b$ In general $U^*_n = a^n\left[\frac{c}{c-n}\right]^b$	$t_U = U'_1 = a\left[\frac{c}{c+1}\right]^b$ $U_2 = U'_2 - U'^2_1$ $U_3 = U'_3 - 3U'_2\,U'_1 + 2U'^3_1$ where $U'_2 = a^2\left[\frac{c}{c+2}\right]^b$ $U'_3 = a^3\left[\frac{c}{c+3}\right]^b$ In general $U'_n = a^n\left[\frac{c}{c+n}\right]^b$
Variation C	$C_v = \sqrt{\frac{c}{b(b+c+1)}}$	$C_v = \sqrt{U_2}\,/\,U_1$	$C_v = \sqrt{U_2}\,/\,U_1$	$C_v = \sqrt{U_2}\,/\,U_1$

Skewness C_s	$C_s = \frac{2(c-b)}{b+c+2}\sqrt{\frac{b+c+1}{bc}}$	$C_s = U_3 / U_2$	$C_s = U_3 / U_2$	$C_s = U_3 / U_2$
Parameter estimation by the method of moments	$b = \frac{1 - \lambda(1 + C_v^2)}{C_v^2}$ $c = \frac{1-\lambda}{\lambda} b,\ a = t_U / \lambda$ where $\lambda = \frac{2C_v - C_s}{4C_v - C_s(1-C_v^2)}$	By numerical solution (e.g. Newton-Raphson method) of the above equations of moments.	By numerical solution (e.g. Newton-Raphson method) of the above equations of moments.	By numerical solution (e.g. Newton-Raphson method) of the above equations of moments.
Time to peak (mode)	$t_p = \frac{a(b-1)}{b+c-2}$	$t_p = a - a\left[\frac{b-1}{bc-1}\right]$	$t_p = a \exp\left[\frac{b-1}{c+1}\right] - a$	$t_p = a \exp\left[-\frac{b-1}{c-1}\right]$
Remarks	Definition of B-function $B(b,c) = \int_0^1 x^{b-1}(1-x)^{c-1}\,dx$ $= \Gamma(b)\Gamma(c) / \Gamma(b+c)$ Definition of the incomplete β-function $\beta(y,b,c) = \int_0^y x^{b-1}(1-x)^{c-1}\,dx$		Definition of U_n^* (= nth moment about the point $t = -a$) $U_n^* = \int_0^\infty (t+a)^n\, u(t)\, dt$	

Note: The above given range for the parameters corresponds to the case that $u(t)$ is bell-shaped.

5. *Beta (B)* (double-bounded/three-parameter)
Beta density function with an extra scale parameter, $a = T_0$, can give a nice form for the SIUH representation, that is

$$u(t) = \frac{(t/a)^{b-1}\,(1-t/a)^{c-1}}{aB(b,c)}, \quad 0 \leqslant t \leqslant a. \tag{16}$$

6. *Double-Power (DP)* (double-bounded/three-parameter)
This term is used to describe the following three-parameter function

$$s(t) = [1 - (1-t/a)^b]^c, \quad t \geqslant 0. \tag{17}$$

The simplicity of the SSC analytical expression, as well as the one for SIUH (see Table II) is remarkable. This form has been extracted from a similar CDF suggested by Kumaraswamy (1980).

7. *Shifted Log-Pearson III (SLP)* (left-bounded/three-parameter)
The usual logarithmic transformation, ($x = \ln t$) applied to the Pearson type-III distribution,

$$f(x) = \frac{c^b(x-d)^{b-1}\,e^{-c(x-d)}}{\Gamma(b)}, \quad x \geqslant d \tag{18}$$

is not proper for the SIUH expression, since $t = e^x$ ranges in $[e^d, \infty)$. In order to decrease the lower bound to zero, we apply the following shifted logarithmic transformation

$$x = \ln(t + e^d)$$

and get

$$u(t) = \frac{c^b}{a\Gamma(b)}\,\frac{[\ln(t/a+1)]^{b-1}}{(t/a+1)^{c+1}}, \quad t \geqslant 0. \tag{19}$$

where $a = e^d$ is a scale parameter.

8. *Minus Log-Pearson III (MLP)* (double-bounded/three-parameter)
This form also originates from the Pearson type-III distribution, by applying the minus logarithmic transformation, i.e. $x = -\ln t$, which gives

$$u(t) = \frac{c^b}{a\Gamma(b)}\,(t/a)^{c-1}[-\ln(t/a)]^{b-1}, \quad 0 \leqslant t \leqslant a, \tag{20}$$

where $a = e^d$ is a scale parameter.

4. Comparison of the Analytical Forms

Since the definition parameters of the above-described forms (a, b, c) are not

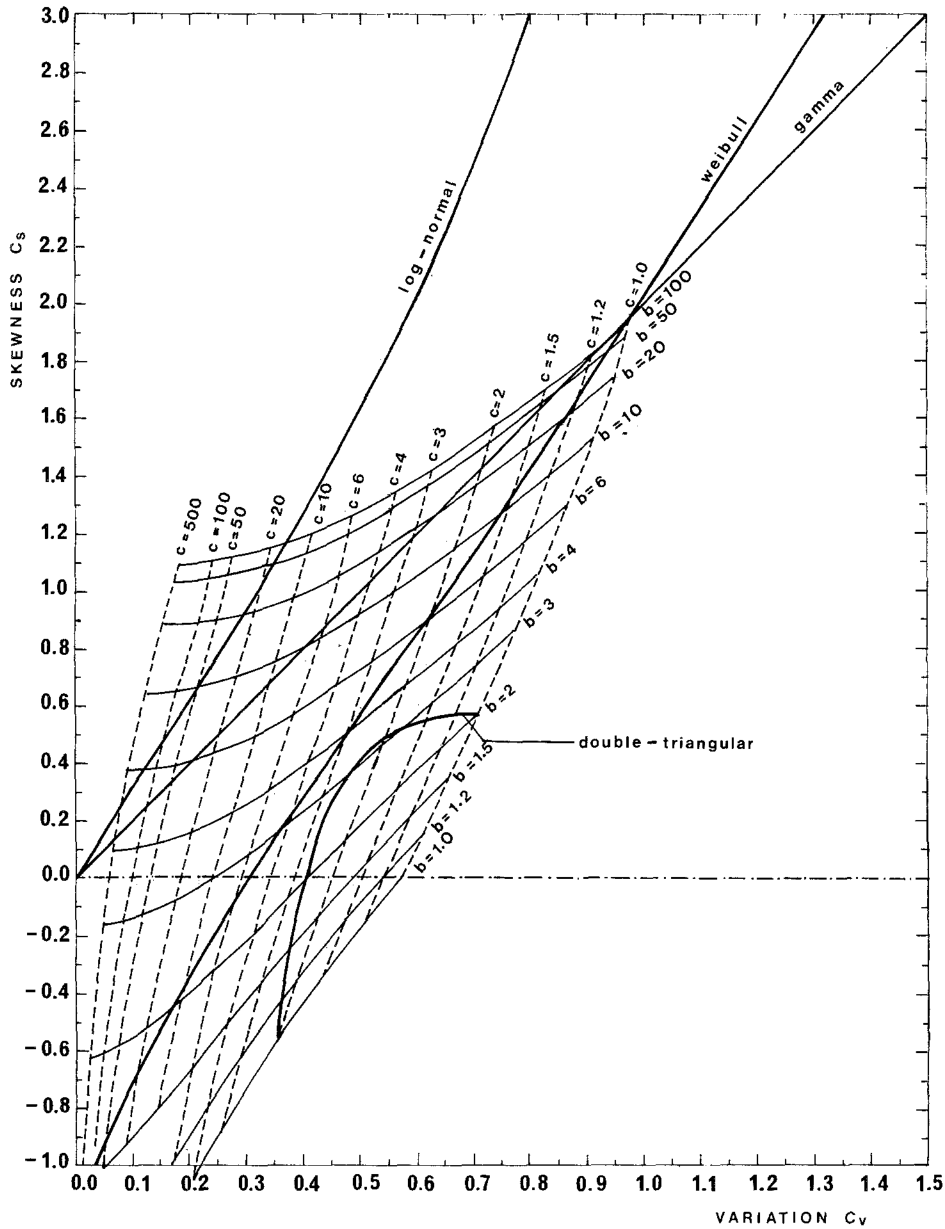


Fig. 1. Skewness coefficient versus coefficient of variation, for the two parameter forms. Relation between definition parameters and coefficients of variation and skewness, for the double-power (DP) form.

comparable, it is preferable to use the first three moments (t_U, U_2, U_3) instead, which can be expressed in terms of the definition parameters (Tables I and II). The derivative dimensionless descriptors

coefficient of variation $C_v = (U_2)^{1/2} / t_U$

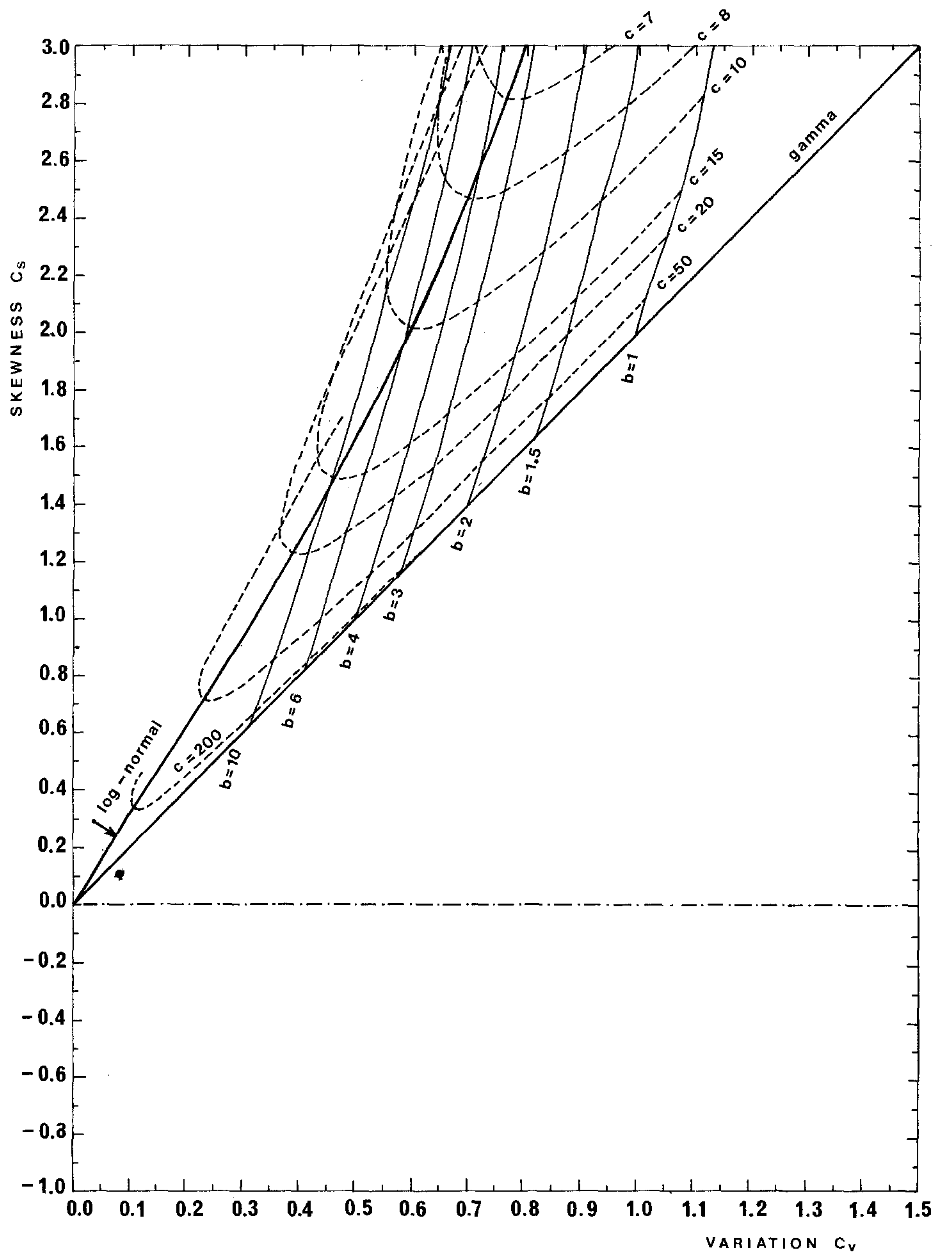


Fig. 2. Relation between definition parameters and coefficients of variation and skewness, for the shifted log-Pearson (SLP) form.

and

skewness coefficient $\quad C_s = U_3 \,/\, (U_2)^{3/2}$

are the best indicators for the comparison.

The two-parameter forms have a fixed relation between C_v and C_s, i.e.

$$C_s = 2C_v \tag{21}$$

for the gamma form and

$$C_s = 3C_v + C_v^3 \tag{22}$$

for the log-normal form, while the relations for the other two forms have no simple analytical expressions. All four relations are plotted in Figure 1, from which we conclude that the double triangular form gives the lowest value of C_s for a given value of C_v, and the log-normal gives the highest.

The domain of C_v and C_s for the three-parameter forms differs from one form to another. In particular, in the case of the beta form, it is easily shown that this domain extends below curve (21) corresponding to the gamma form. In the shifted log-Pearson form, the domain extends above curve (21) and exceeds curve (22), corresponding to the log-normal form (see Figure 2). In the double-power form, the domain is quite similar to that of the beta form, but with an extension above curve (21) (see Figure 1). Finally, the minus log-Pearson form has the widest domain, extending below curve (22) (see Figure 3). The above observations and Figures 1 to 3 are quite helpful for the selection of the ideal form.

Systematic examination of the various SIUH shapes for specified values of the first three moments (or parameters t_U, C_v and C_s), showed that, generally, the shapes are quite similar. Figures 4 and 5 illustrate the variation of the two main characteristics (the time to peak and the peak discharge) of the SIUH and the DUH for the duration $D=t_U$, respectively, versus the variation of coefficients C_v and C_s. Due to the similarity of the various shapes, it is difficult to distinguish the curves for each separate form. Thus, in most cases, one single curve for each value of C_v has been drawn in Figures 4 and 5. This curve represents all the three-parameter forms. The characteristics of the two-parameter forms are also in agreement with these curves. An exception to this is the double-triangular form, yielding to a deviating higher peak of SIUH, due to the discontinuity in its derivative. The deviation, however, reduces in the case of DUH, which is of more practical interest.

The above discussion shows that all the examined forms are of a similar performance for the SIUH representation. It is obvious that the three-parameter forms are more adjustable, while the two-parameter forms are simpler. The selection of a specific form may be based on the values of the descriptors C_v and C_s (see section 5. The simplicity of the form may also be considered. We note that the double-triangular, Weibull and double-power forms have simpler expressions for both $u(t)$ and $s(t)$ functions. Calculations do not require the use of computers or statistical tables. From the other side, the double-triangular, gamma, log-normal and beta forms have simpler relations between their moments and their definition parameters. This is of interest when parameters are estimated from moments.

A final observation at this point drawn from Figure 5, is that the magnitude

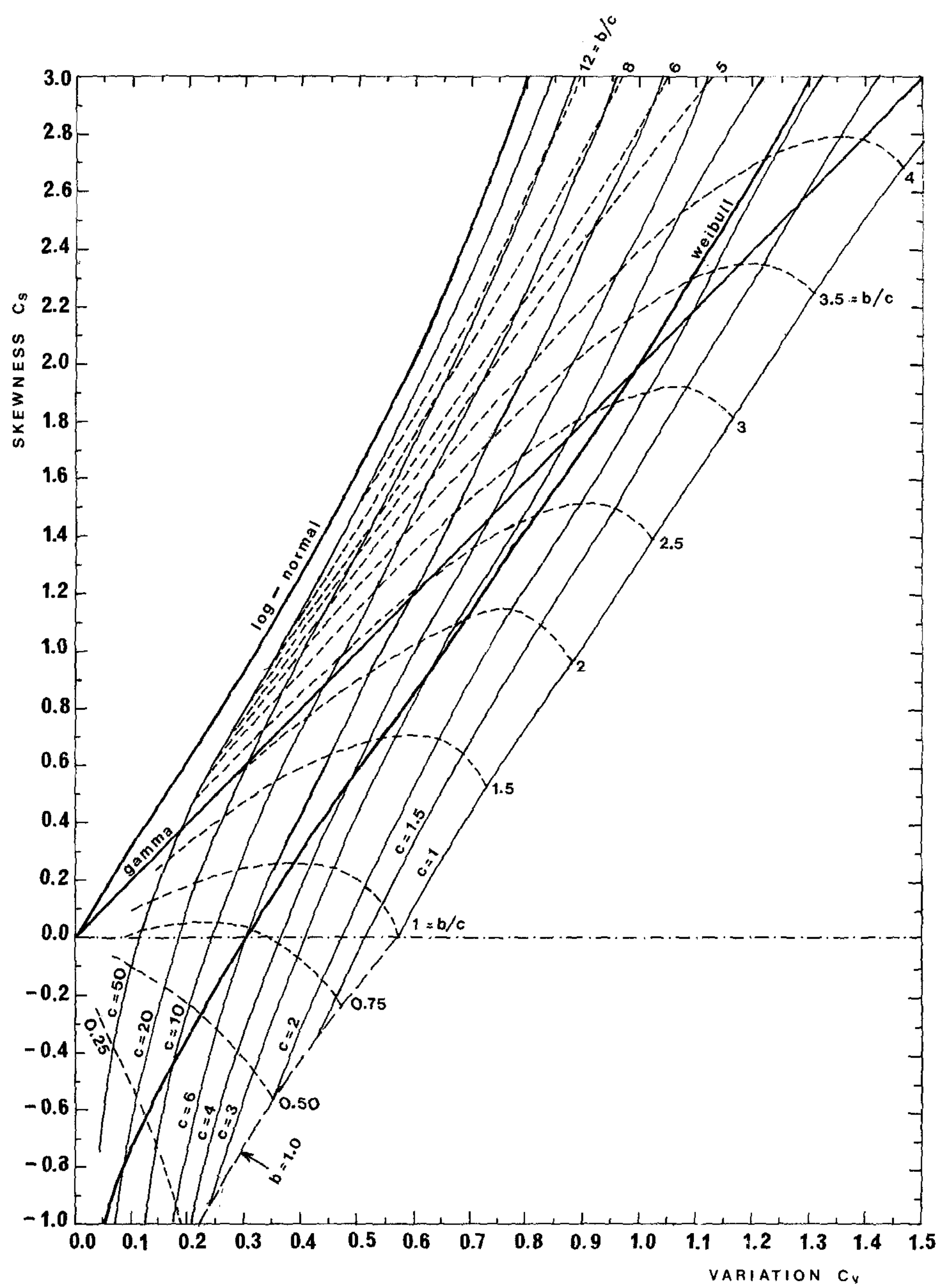


Fig. 3. Relation between definition parameters and coefficients of variation and skewness, for the minus log-Pearson (MLP) form.

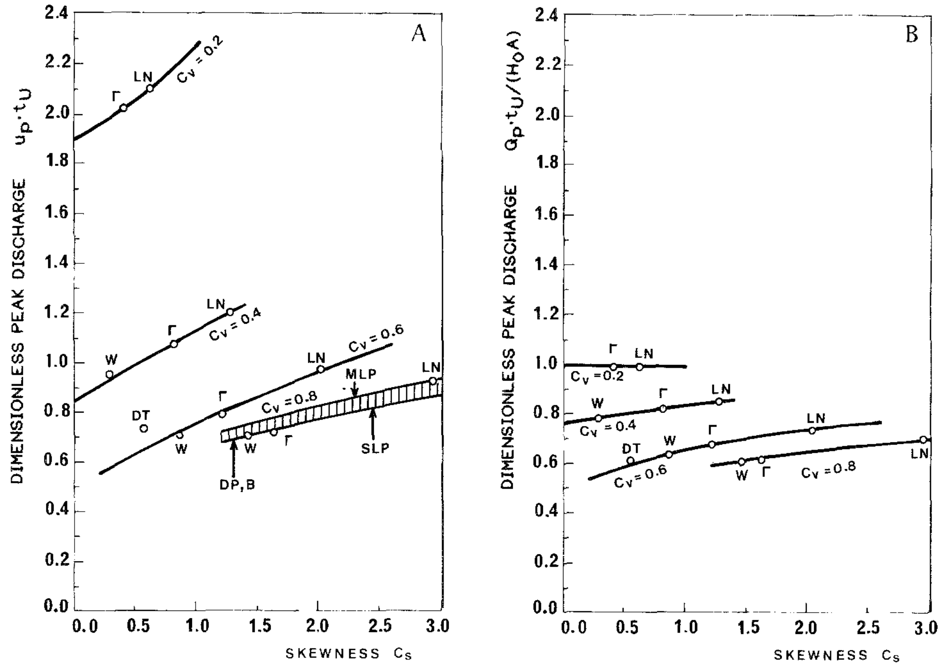


Fig. 4. Dimensionless peak discharge of (A) standardized instantaneous unit hydrograph (SIUH) and (B) unit hydrograph with duration $D = t_U$ versus the coefficients of variation and skewness, for various analytical forms.

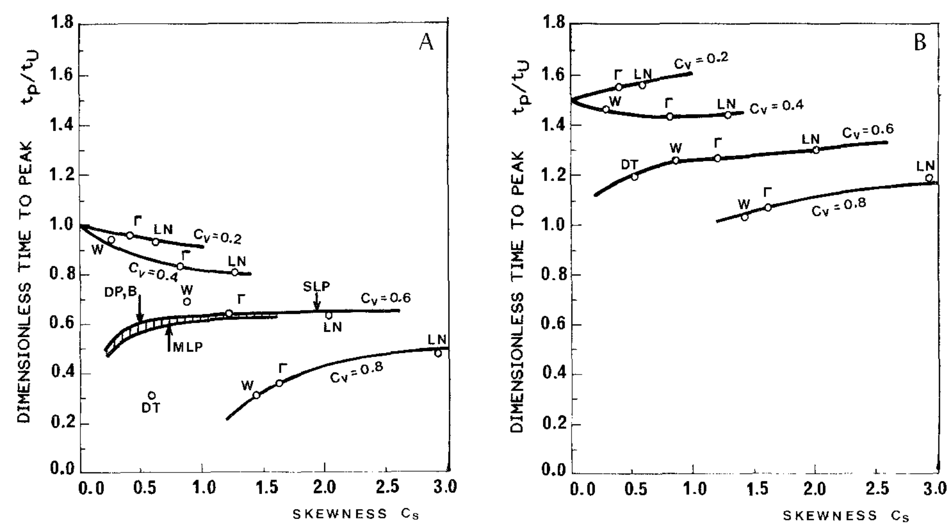


Fig. 5. Dimensionless time to peak of (A) standardized instantaneous unit hydrograph (SIUH) and (B) unit hydrograph with duration $D = t_U$ versus the coefficients of variation and skewness, for various analytical forms.

of the peak discharge of the IUH or DUH increases with the decrease of the coefficient of variation as well as with the increase of the skewness coefficient.

5. Parameter Estimation by the Method of Moments

We assume that the IUH identification is based on recorded rainfall and runoff data of the catchment and not on various catchment characteristics.

Let $I(t)$ be the net hyetograph of a recorded rainfall event and $Q(t)$ is the corresponding surface runoff hydrograph derived from the recorded data. Dividing the above functions with the total net rainfall depth, H, and the total surface runoff volume, V, respectively, we get the *standardized net hyetograph i(t)* and the *standardized surface runoff hydrograph* $q(t)$. Furthermore, let t_I and t_Q be the times from the origin to the centers of area of $i(t)$ and $q(t)$, respectively, and I_n and Q_n on the n^{th} central moments of $i(t)$ and $q(t)$.

Nash (1959) showed that the moments of SIUH are related to the above moments by

$$t_U = t_Q - t_I, \tag{23}$$

$$U_2 = Q_2 - I_2, \tag{24}$$

$$U_3 = Q_3 - I_3. \tag{25}$$

Somewhat more complex relations exist for moments of higher orders.

These relations permit a simple calculation of the SIUH moments which can then determine the definition parameters of a specific SIUH form.

The whole process of the SIUH identification using this method, consists of the following steps:

(1) Calculate the moments of $i(t)$ and $q(t)$;
(2) calculate the moments of $u(t)$ using (23) through (25);
(3) calculate the descriptors C_v and C_s;
(4) select an analytical form, which is proper for C_v and C_s;
(5) calculate the definition parameters of the selected form.

The equations in Tables I and II, relating the SIUH's moments and descriptors to the definition parameters, should be used for step 5 of the process. Double-triangular, gamma, log-normal and beta forms are the simplest to be used with this method. The other analytical forms require the use of a numerical procedure for the solution of equations. The nomographs of Figures 1, 2 and 3 support the form selection in step 4 and may replace the numerical methods of equation solving in step 5, when there is no need for high accuracy.

The real problem with this method is that the SIUH's moments calculated from separate recorded rainfall/runoff events usually differ to a remarkable degree. One solution to that problem, oriented towards the derivation of a mean unit hydrograph, is obtained by taking the average of each moment. The obtained parameters of the SIUH can then be modified in order to get more severe unit hydrographs, to be used in design floods.

6. Parameter Estimation by the Method of Least Integral Square Error

The method of moments is the simplest parameter estimation procedure, but it is not the only obtainable one, since other methods could also be set up for this aim.

The method proposed here is oriented towards the minimization of an objective function, defined as the integral square error between the recorded runoff hydrograph and the derived, with the use of the selected analytical form, convoluted runoff hydrograph. This objective function can be formulated as

$$g(a, b, c) = \sum_{i=1}^{m} (Q_i - Q^*_i)^2, \tag{26}$$

where $g(\ )$ is the objective function, a, b, c are the parameters of the SIUH, which in this point are considered as decision variables (the method can be applied for more than three parameters, as well), i is a time index, Q_i is the ordinate of the recorded surface runoff hydrograph, Q^*_i is the corresponding ordinate of the convoluted surface runoff hydrograph and m is a sufficiently large integer constant.

We note that $g(\ )$ is a convex function and neither itself nor its derivatives can have simple analytical forms. Thus, the analytical optimization methods cannot be applied here. The minimization of $g(\ )$ is carried out through a proper iterative numerical procedure. In each iteration, a set of values of the parameters is assumed and the value of the objective function is calculated, as described in the following four steps:

(1) Calculate SIUH and SSC for the assumed set of parameters;
(2) calculate the unit hydrograph for the appropriate rain duration, using (5) and (3);
(3) calculate the flood hydrograph by convolution of the unit hydrograph and the recorded net hyetograph;
(4) calculate the integral error betwen recorded and convoluted flood hydrographs, by (26).

A fully general algorithm, in **Pascal** programming language, has been developed for the above minimization procedure, which systematically executes the required iterations. It seems like the *bisection algorithm* used for the equation solving, but uses three successive points of each (decision) variable, in the way that the middle point corresponds to the lowest value of $g(\ )$. There are no restrictions on the number of decision variables, but the addition of more variables exponentially increases the required number of computations. Thus, the algorithm is time-consuming, but its generality is considerable. In the examined problem, the use of double-triangular, Weibull, or double-power forms speeds up computations because of the simplicity of the functions $s(t)$ in these forms.

This method produces better results than the method of moments, and this will be verified later on. The method can be easily extended to the case where more than one recorded flood hydrograph is available. In this case, the objective function should be defined as the total error for all hydrographs.

7. Advantages of the Parametric Approach

As was pointed out in the introduction, the parametric approach is, in general, less accurate than the standard linear approach, but it has certain advantages. The first advantage is its simple and secure numerical computations (though the mathematical background may seem somewhat complicated).

Not only is the limited number of parameters a handicap, but it can also be advantageous in some cases, especially when we need to establish the relationship between the unit hydrograph and catchment characteristics. Such a relationship assists the unit hydrograph derivation in neighbouring ungauged catchments.

The preselection of a smooth analytical form for the representation of IUH, guarantees a smooth unit hydrograph and S-curve. Thus, the parametric method can also be applied to the smoothing of a DUH, derived by the usual linear method, in order to avoid the problems that frequently appear when a DUH is converted from one duration to another (negative ordinates, unexpected successive peaks etc.).

Finally, the main advantage of the parametric approach is the possibility of applying it to large catchments where the usual linear method fails.

8. Application Examples from Case Studies

The first example is taken from the study of the Thessalia Basin, middle Greece (Xanthopoulos *et al.*, 1988). One of the aims of the study was the derivation of design floods in several sites of the basin. Because of the inadequate equipment at the basin, the unit hydrograph derivation has been based mainly on the data of one sub-basin of the Pinios river (Sarakina) with an area of 1061 km^2. A preliminary investigation of recorded flood hydrographs and simultaneous charts of the two rainfall recorder stations of the sub-basin, produced six flood events which were suitable for analysis. The large catchment area and the inadequate number of rain recording stations inhibited the use of the usual linear method for the unit hydrograph derivation. Furthermore, the parametric method was superior in the examined problem, because of the need to transfer Sarakina's unit hydrograph to other sites of the basin.

The method of moments was used for the parameter estimation. The moments of SIUH were computed by Equations (23) to (25). As shown in Table III, moments

Table III. SIUH moment from six flood events at Sarakina Basin

SIUH moment	Min. value	Max. value	Average value
t_U (h)	3.50	9.75	6.84
U_2 (h^2)	9.15	24.42	14.78
U_3 (h^3)	6.34	47.95	31.66

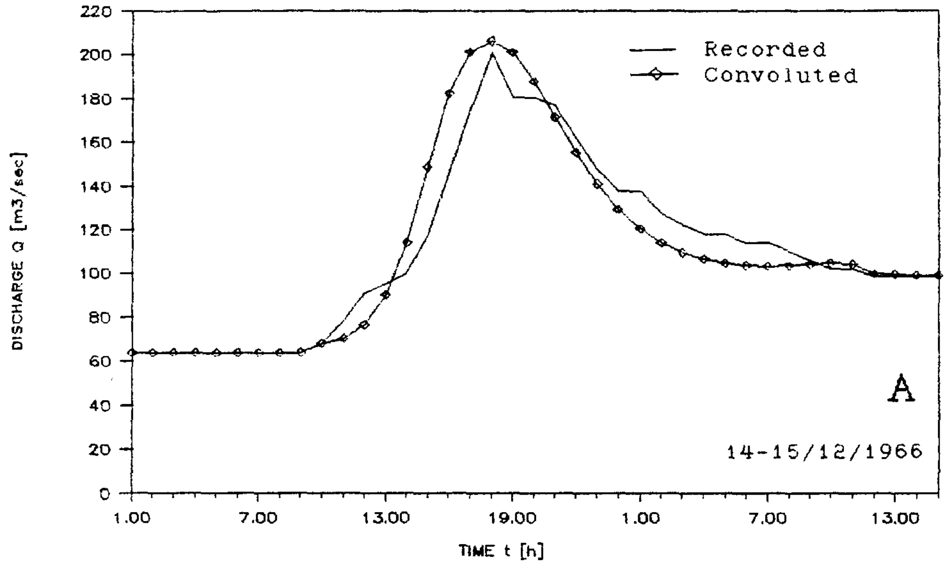
computed from separate events display large deviations amongst them, apparently due to the unmeasured spatial nonuniformity of rainfall over a large catchment area.

The coefficient of variation, calculated from the average moments, is C_v =0.563 and the skewness coefficient C_s = 0.557. These values normally support the selection of the double-triangular or Weibull form (see Figure 1), but because of the major severity of the design floods, the log-normal form was finally selected. The parameters of that form, calculated by the relations of Table I are b = 0.55 and a = 1.79. The flood hydrographs of the six events were reconstructed by convolution of the 1-hour DUH (obtained by (5) and (3)), and the related hyetographs. Comparisons between them and the corresponding recorded hydrographs gave relatively satisfactory results, the deviations being unavoidable because of the large ranges of the SIUH's moments. Figure 6 illustrates two of these comparisons concerning the cases of minimum and maximum deviations.

A second example is oriented towards the comparison between unit hydrographs derived by each of the above parameter estimation methods. The data in this example come from the study of Ajak stream, northern Greece (Koutsoyiannis *et al.*, 1982). The catchment area upstream of the examined Iliolousto dam site is 252 km^2. The unit hydrograph was derived by the usual linear method from four recorded flood events. One of these events (14–15 August 1981) was used here to develop this example. The moments of SIUH, computed from this event by (23) through (25) are t_U = 5.30 h, U_2 = 5.02 h^2 and U_3 = 13.04 h^3, which give C_v = 0.423 and C_s = 1.159. The double power form is suitable for these values and has been selected. The parameters calculated by the method of moments are a = 116.9, b = 61.0 and c = 9.15. The parameters calculated by the least-square error method are quite different: a = 203.9, b = 160.0 and c = 25.75. The square error is 188.2 for the first case and 41.6 for the second. The 1-h DUHs for both cases are plotted in Figure 7, in comparison with the one derived by linear analysis. This figure shows that the method of moments underestimates the peak flow and that the least-square error method is superior.

9. Conclusions

(1) The parametric approach gives a good approximation to a catchment's unit hydrograph, although it uses a limited number of parameters. The uncertainties



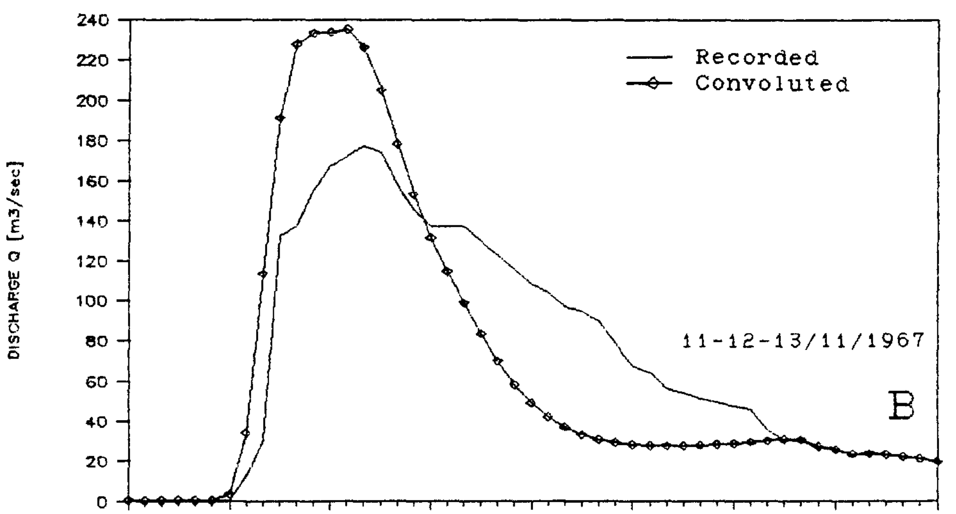


Fig. 6. Comparison between recorded flood hydrographs at Pinios, site Sarakina, and convoluted hydrographs, using the log-normal form, concerning the events with (A) minimum and (B) maximum deviation. Parameters were calculated by the method of moments.

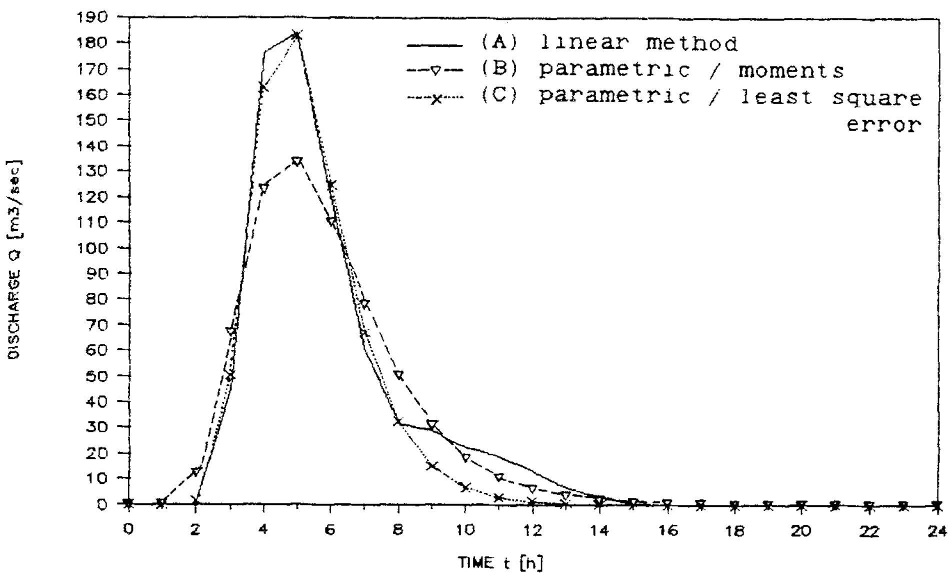


Fig. 7. Comparison between 1-h unit hydrographs of Ajak stream at Iliolousto site, derived by (A) the usual linear method, (B) the parametric approach with the method of moments, and (C) the parametric approach with the least-square error method. All unit hydrographs have been derived from one flood event, which occurred on 14–15 August 1981.

of this approach are mainly due to different catchment behaviour in different flood events, as well as inaccuracies in preliminary data processing (such as baseflow and rainfall losses separation), and secondary due to the limited number of parameters used.

(2) The advantages of this approach are the simple numerical computatons required for the UH identification and its ability to be applied to large gauged catchments, where the standard method fails. The approach assists the unit hydrograph derivation in ungauged catchments.

(3) Eight different suitable analytical forms for the instantaneous unit hydrograph are presented in this paper, accompanied by complete analytical expressions and nomographs required for the application. Two different methods of parameter estimation are studied, the first being the well-known method of moments, while the second is based on the minimization of the integral error between derived and recorded flood hydrographs.

(4) When the method of moments is used for the parameter estimation, those IUHs with a smaller variation coefficient and larger skewness coefficient, yield higher peak discharges. From the examined two-prameter analytical forms, the log-normal one yields the highest peak discharge, when parameters are calculated from the first two moments. The three-parameter forms give quite similar unit hydrographs, no matter which particular form is used.

(5) The parameter estimation method based on the integral square error gives

more accurate results, particularly in the estimation of the peak discharge, but its application procedure is more complicated than that of the method of moments.

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