Hydrofractals ’03

An international conference on fractals in hydrosciences

Monte Verità, Ascona, Switzerland, 24-29 August 2003

A toy model of climatic variability with scaling behaviour

Demetris Koutsoyiannis

Department of Water Resources, School of Civil Engineering,

National Technical University, Athens, Greece

Introduction: The notion of a toy model

- **Definition**: A model in which the features represented are kept to a minimum in order to show that some empirical phenomenon can or cannot be produced from primitive assumptions (adapted from Cox and Isham, 1998)
- **Objectives**
  - Investigate whether simple mechanisms can produce a complex phenomenon
  - Identify essentials and discard details in the system dynamics
  - Identify sets of parameters for which the phenomenon occurs
- **Parameter issues**
  - A small number of parameters is involved
  - Formal fitting may be irrelevant
- **Examples**
  - ENSO dynamics (Andrade et al., 1995)
  - Biological evolution of species (Wandewalle & Ausloos, 1996)
  - Attraction of parasites and predators (Freund & Grassberger, 1992)

The phenomenon studied: Simple scaling of climatic time series in discrete time

Clarifications

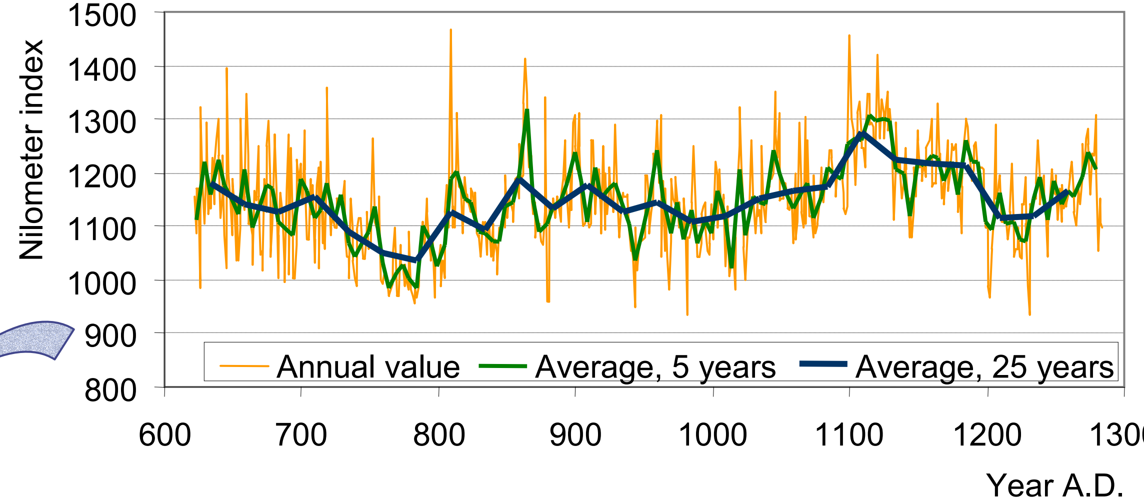
- Scaling is meant here in terms of the behaviour of the time series aggregated (averaged) on different time scales
- Time scales are from annual to thousands of years
- Long time series are required for the study

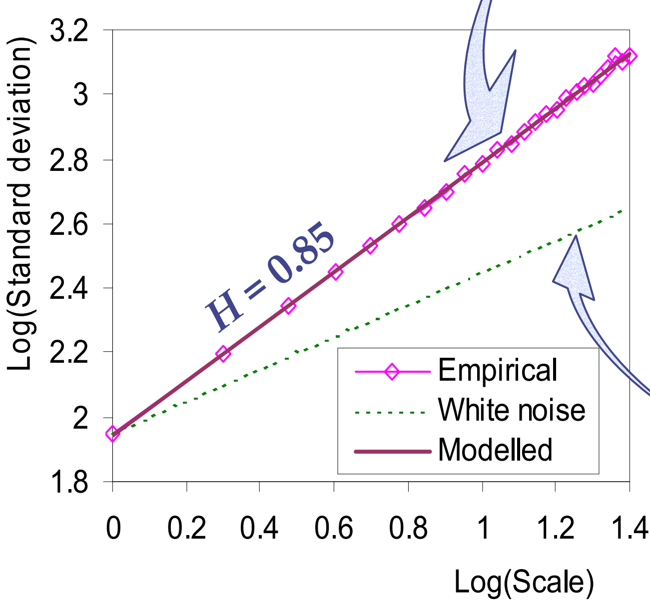
A simple scaling process as a stochastic process

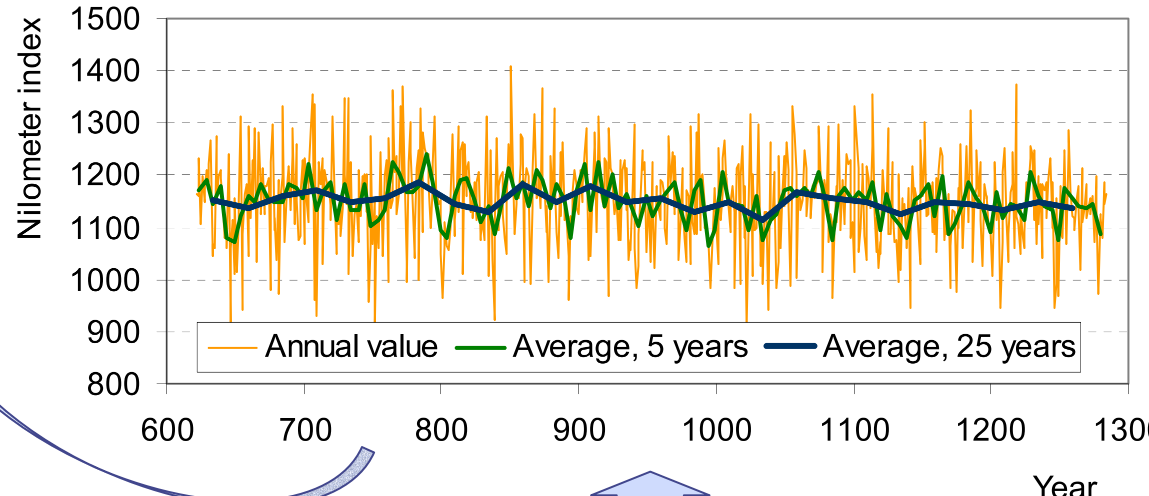
A stochastic process at the annual scale	X_i
The mean of X_i	$\mu := \mathrm{E}[X_i]$
The standard deviation of X_i	$\sigma := \sqrt{\mathrm{Var}[X_i]}$
The lag-j autocorrelation of X_i	$\rho_j := \mathrm{Corr}[X_i, X_{i-j}]$
The aggregated stochastic process at scale $k \geq 1$	$Z_i^{(k)} := \sum_{l=(i-1)k+1}^{ik} X_l$
The mean of $Z_i^{(k)}$	$E[Z_i^{(k)}] = k\,\mu$
The standard deviation of $Z_i^{(k)}$	$\sigma^{(k)} := \sqrt{\mathrm{Var}\,[Z_i^{(k)}]}$
Definition of a simple scaling stochastic process or a simple scaling signal (SSS; also known as (a) stationary increments of self-similar process (b) Fractional Gaussian noise – FGN)	$(Z_i^{(k)} - k\mu) \stackrel{\mathrm{d}}{=} \left(\frac{k}{l}\right)^H (Z_j^{(l)} - l\mu)$ for any scales k and l and for a specified H ($0 < H < 1$) known as the Hurst coefficient
The standard deviation of an SSS $Z_i^{(k)}$ (a power law of scale k)	$\sigma^{(k)} = k^H \sigma$
The lag-j autocorrelation of an SSS $Z_i^{(k)}$ (a power law of lag j; independent of scale k)	$\rho_j^{(k)} = \rho_j \approx H(2H-1)j^{2H-2}$ for $j > 0$

Empirical basis of the study: (a) Nile data set

The Nilometer series indicating the annual minimum water level of the Nile river for the years 622 to 1284 A.D.
(663 years; Beran, 1994)



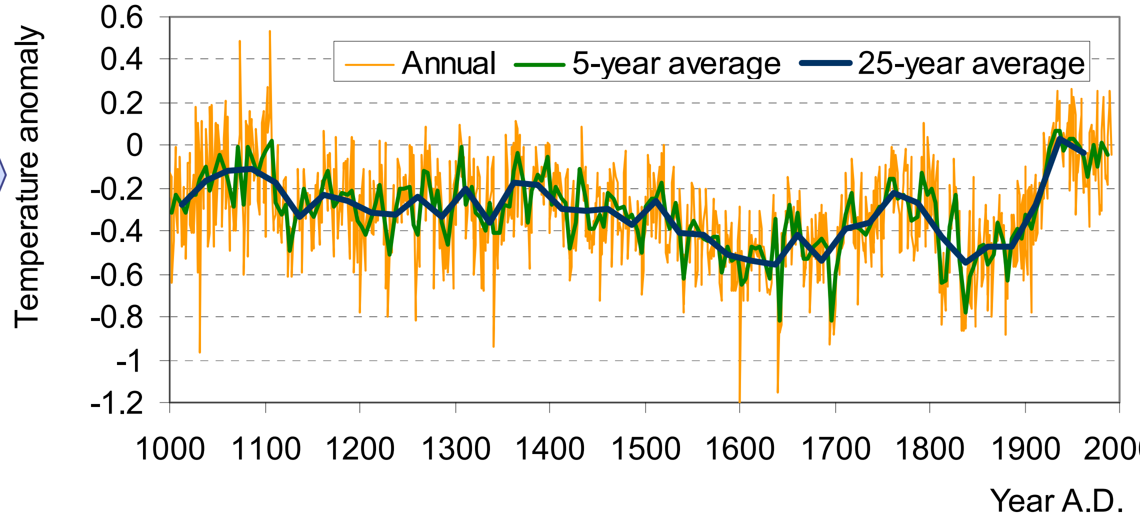




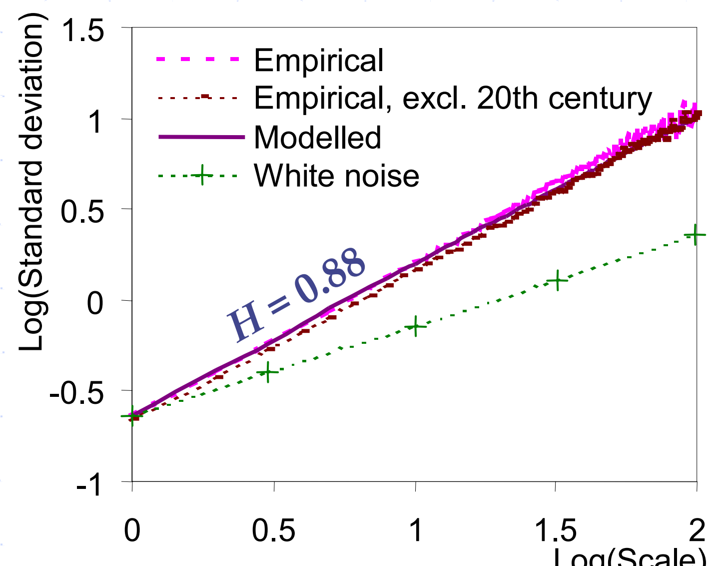
A white noise series (for comparison)

(b) Jones data set

Northern Hemisphere temperature anomalies in °C with reference to 1961–1990 mean (992 years, Jones et al., 1998)



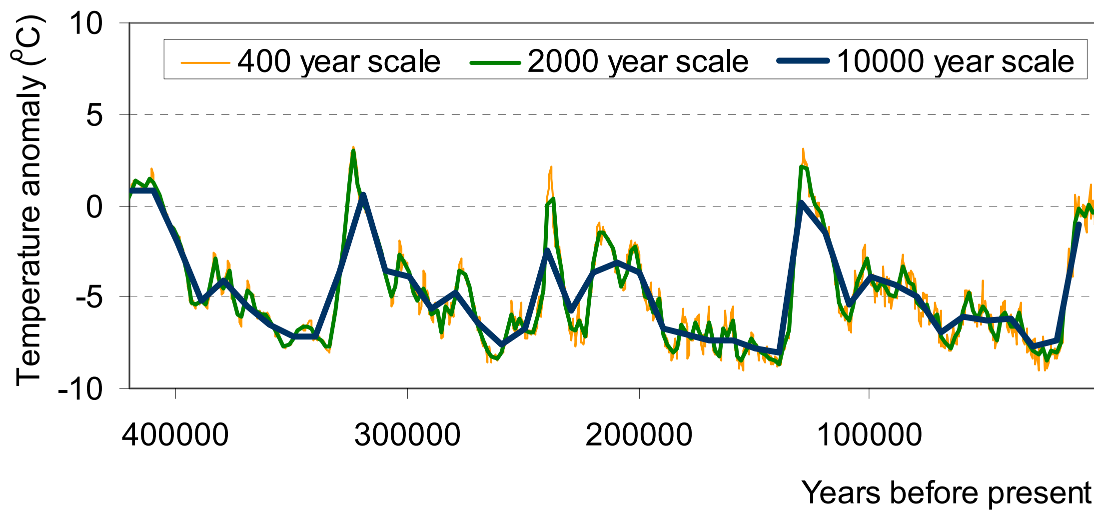
This series was constructed using temperature sensitive paleoclimatic multi-proxy data from 10 sites worldwide that include tree rings, ice cores, corals, and historical documents. Only four of the proxy data series go back before 1400 AD and, therefore, data prior to about 600 years ago are more uncertain.



(c) Vostok data set

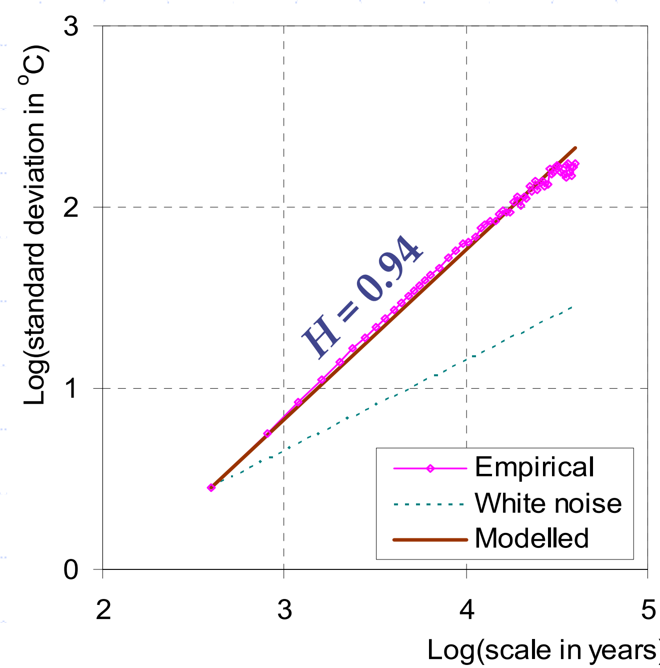
The Vostok ice core deuterium data set going back to 422 766 years before present (Petit et al., 1999)

Temperature difference with reference to the mean recent time value



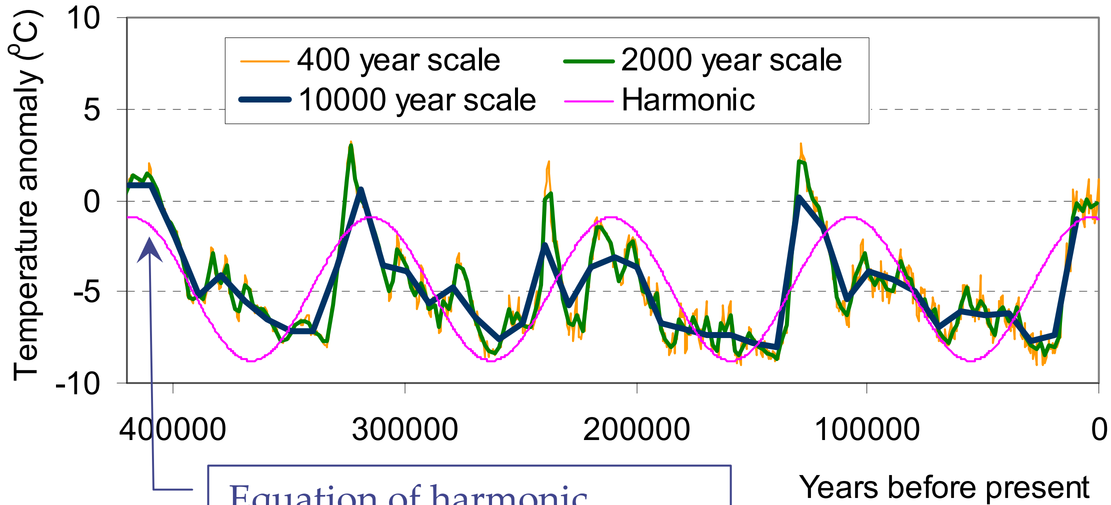
This temperature difference is calculated based on the deuterium content of the ice using a deuterium/temperature gradient of 9‰/°C, after accounting for the isotopic change of sea-water.

The temporal resolution ranges from 17 years (present time) to 631 years. Here the series was re-interpolated using a constant 400 year temporal resolution.



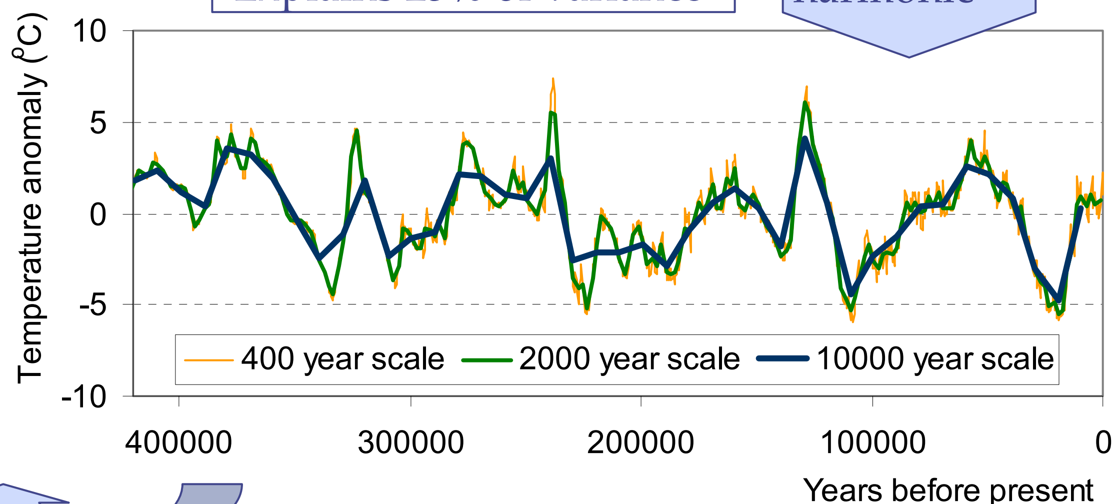
(c_2) Vostok data set adapted

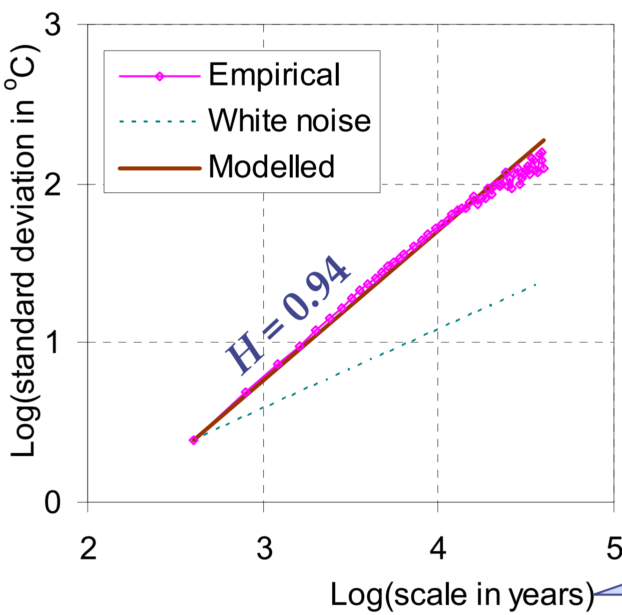
- Vostok data series of temperature difference
- Identification of periodicity
- Plot of the principal harmonic roughly corresponding to the period of orbital stretch



Equation of harmonic
$\widetilde{x}_t = 3.917 \cos(2\pi\, t/\tau + 0.2146) - 4.856$
τ = 103 598 years
Explains 25% of variance

Subtraction of the harmonic

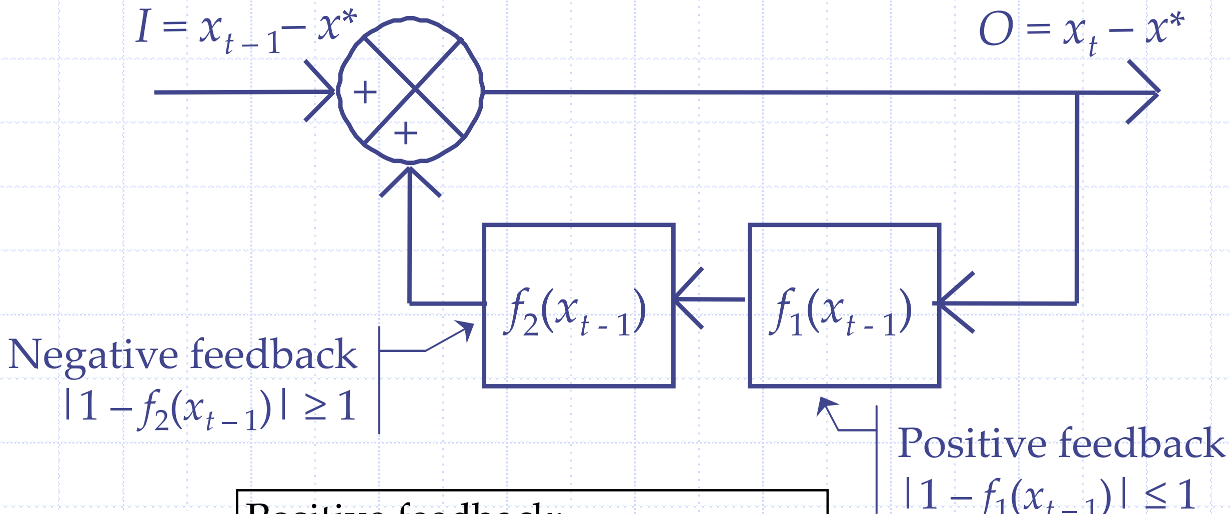


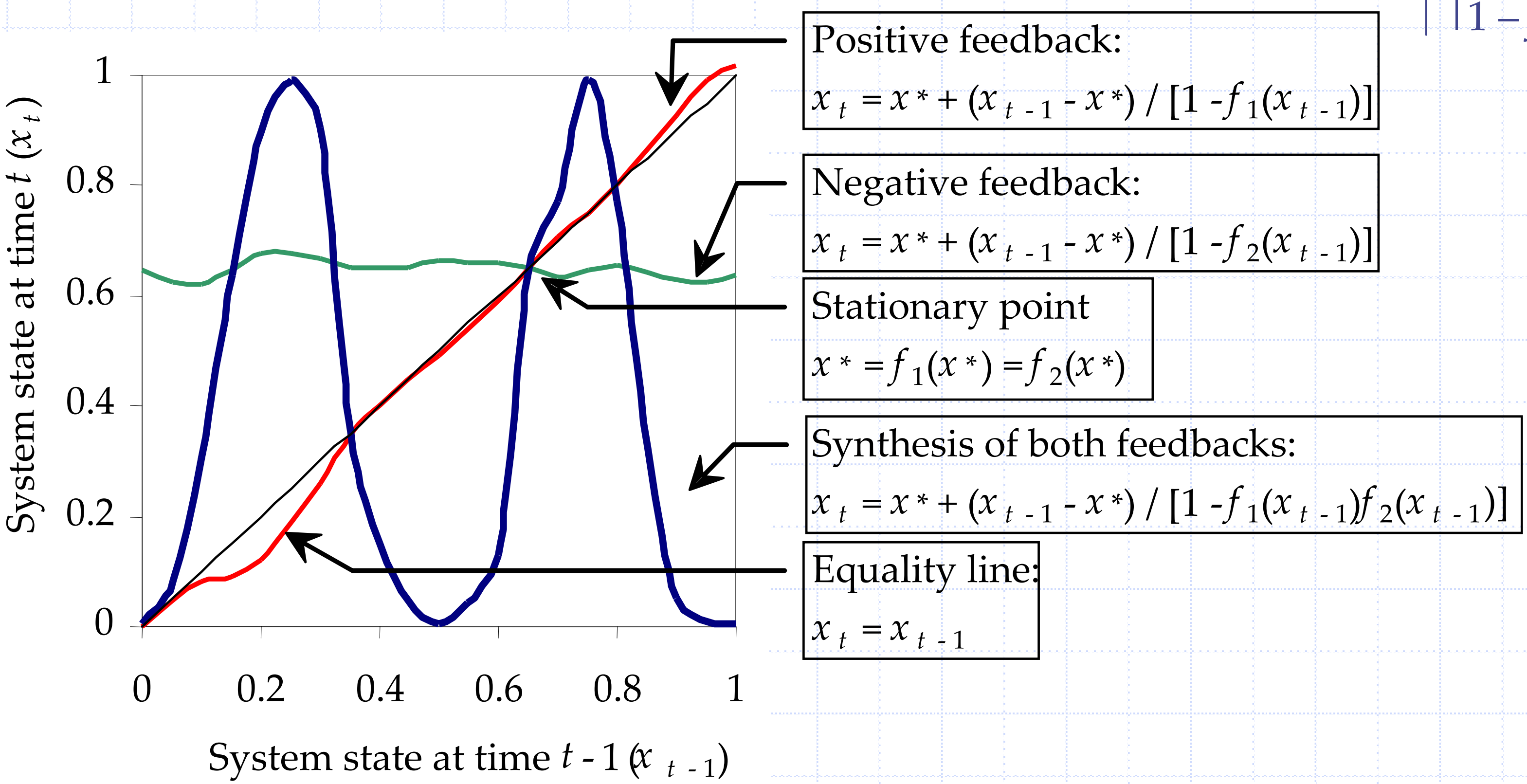


Major climate change processes and feedbacks

- Milankovitch cycles (in scales of thousands of years)
- Ice-albedo feedback (positive)
- Water vapour feedback (positive)
- Cloud feedback (negative)

Synthesis of positive and negative feedbacks





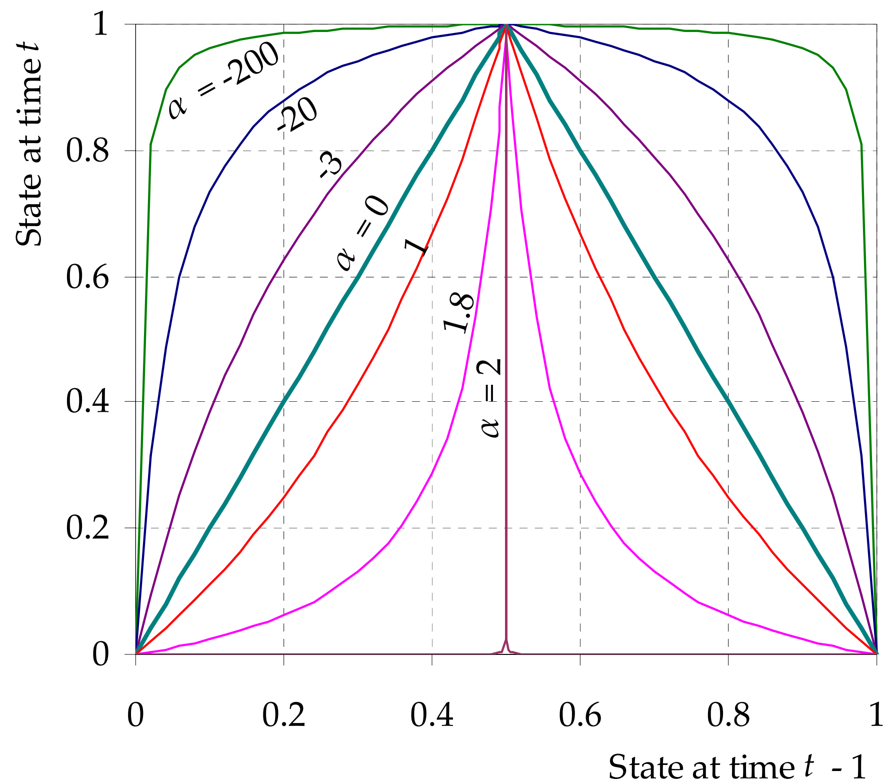
Simplified modelling of compound positive and negative feedbacks

- Starting point:
  The generalised tent map

$$x_t = g(x_{t-1}; \alpha) = \frac{(2-\alpha)\min(x_{t-1}, 1-x_{t-1})}{1-\alpha\min(x_{t-1}, 1-x_{t-1})}$$

with $0 \le x_t \le 1, \ \alpha < 2$

- Example usage: The map approximates the relation between successive maxima in the variable $x(t)$ from the Lorenz equations that describe climatic dynamics (Lasota and Mackey, 1994, p. 150)



Simplified modelling of compound positive and negative feedbacks (2)

- More complex maps resulting from the generalised tent map

$$x_t = g_n(x_{t-1}; \alpha) = \underbrace{g(g(\ldots(g}_{n}(x_{t-1}; \alpha)\ldots); \alpha); \alpha)$$

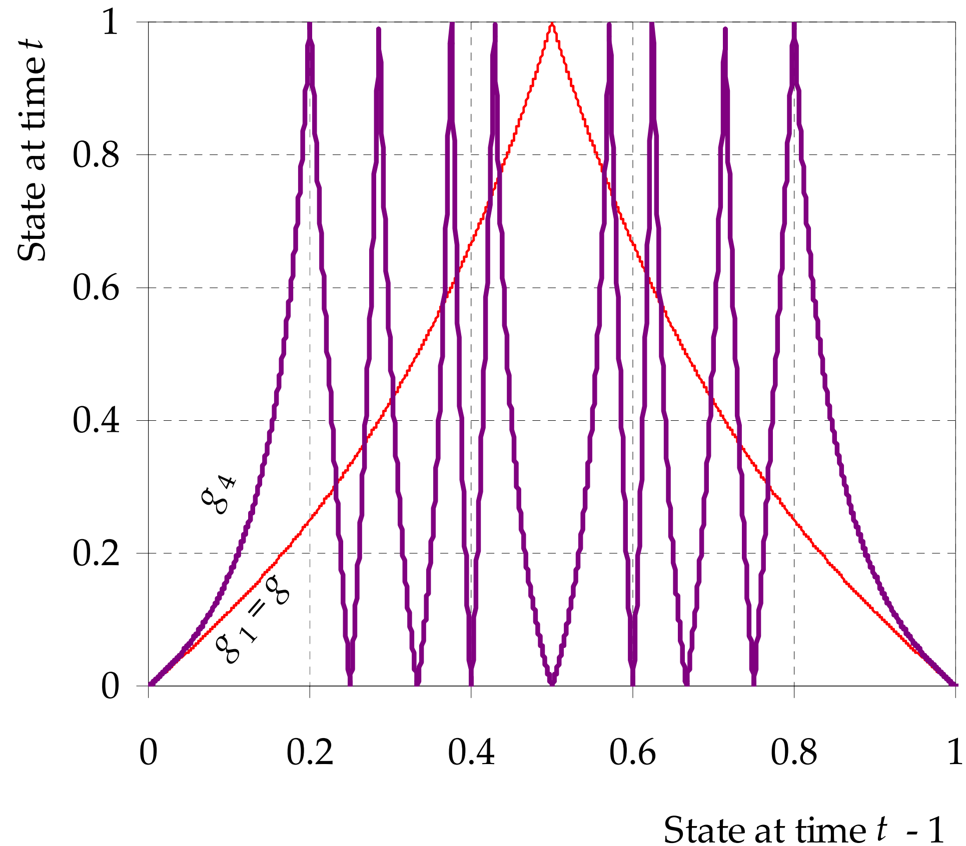
- Equivalent definition of $g_n(\,)$

$x_t = y_{n\,t}$ with $y_{n\,t} = g(y_{n\,t-1}; \alpha)$, $y_0 = x_0$, $t = 0, 1, 2, \ldots$

where the intermediate terms

$y_{(n-1)t}, \ldots\; y_{n\,t-1}$

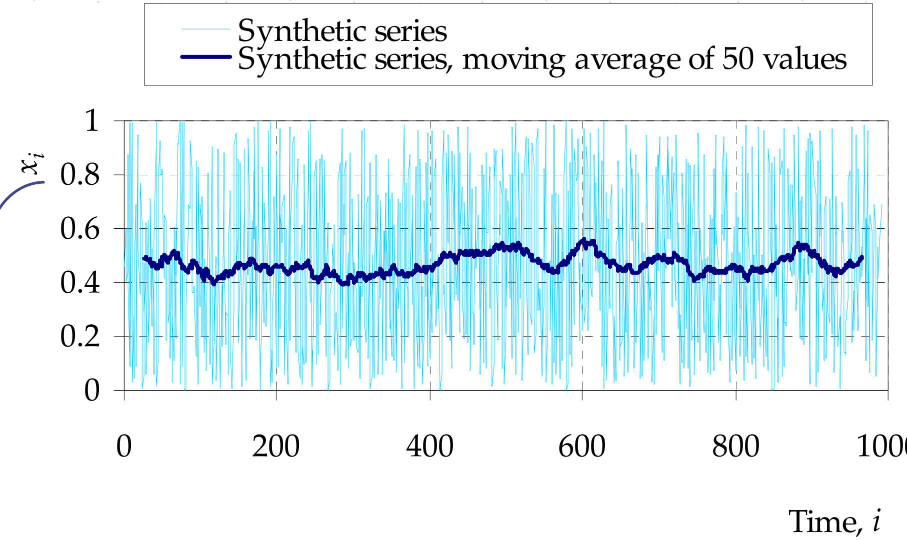
are regarded as *hidden terms*

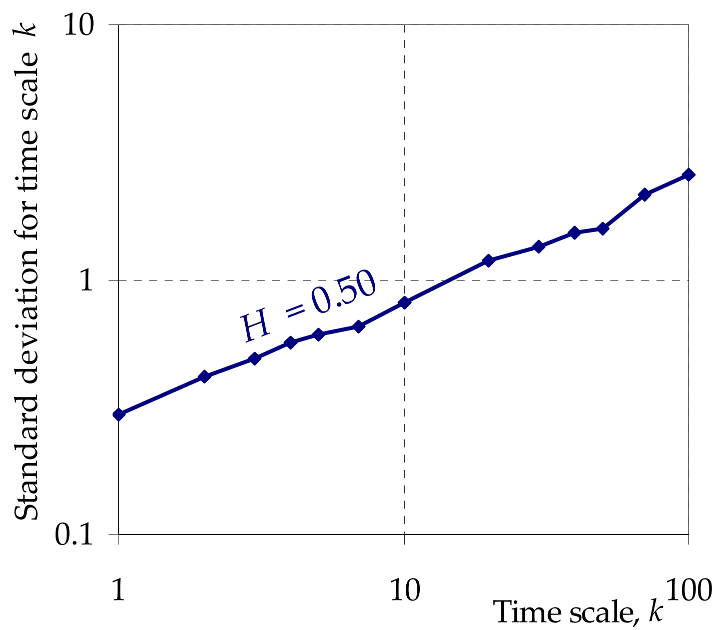


Resulting time series

A time series generated by the transformation $g_4(x; \alpha)$ with $\alpha = 0.317$

- Random appearance at the basic scale
- Stable behaviour at larger scales
- Hurst coefficient = 0.5





The toy model

- Make parameter of the tent transformation time dependent using the same (tent) transformation

  $z_t = G(z_{t-1}; \kappa, \lambda) = g(z_{t-1}; \kappa\, \alpha_{t-1})$ with $\alpha_t = g(\alpha_{t-1}; \lambda)$

- Extend the tent transformation by adding hidden terms

  $z_t = G_n(z_{t-1}; \kappa, \lambda)$

  defined by

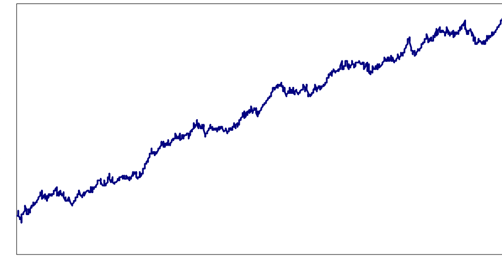
  $z_t = y_{n\,t}$ with $y_{n\,t} = G(y_{n\,t-1}; \kappa, \lambda)$, $y_0 = z_0$, $t = 0, 1, 2, \ldots$

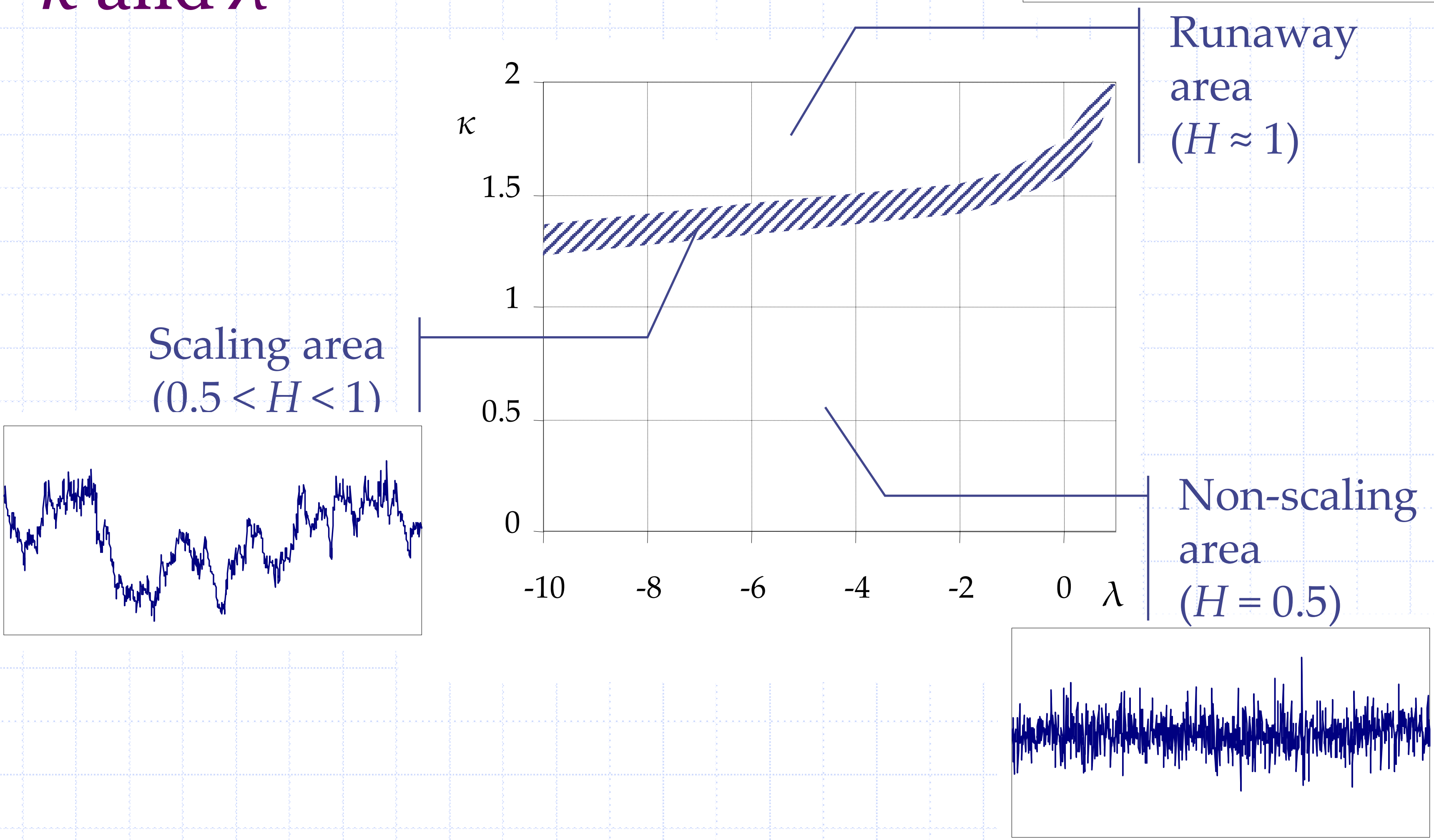
- Apply a rescaling the transformation to shift from [0, 1] to [0, ∞)

  $x_t = b + c \tan(\pi\, z_t / 2)^d$

- The final model for x_t
  - is two dimensional (involves two degrees of freedom corresponding to α_0 and z_0)
  - contains five parameters (κ, λ, b, c, d)

Model behaviour with respect to parameters κ and λ





Parameter fitting

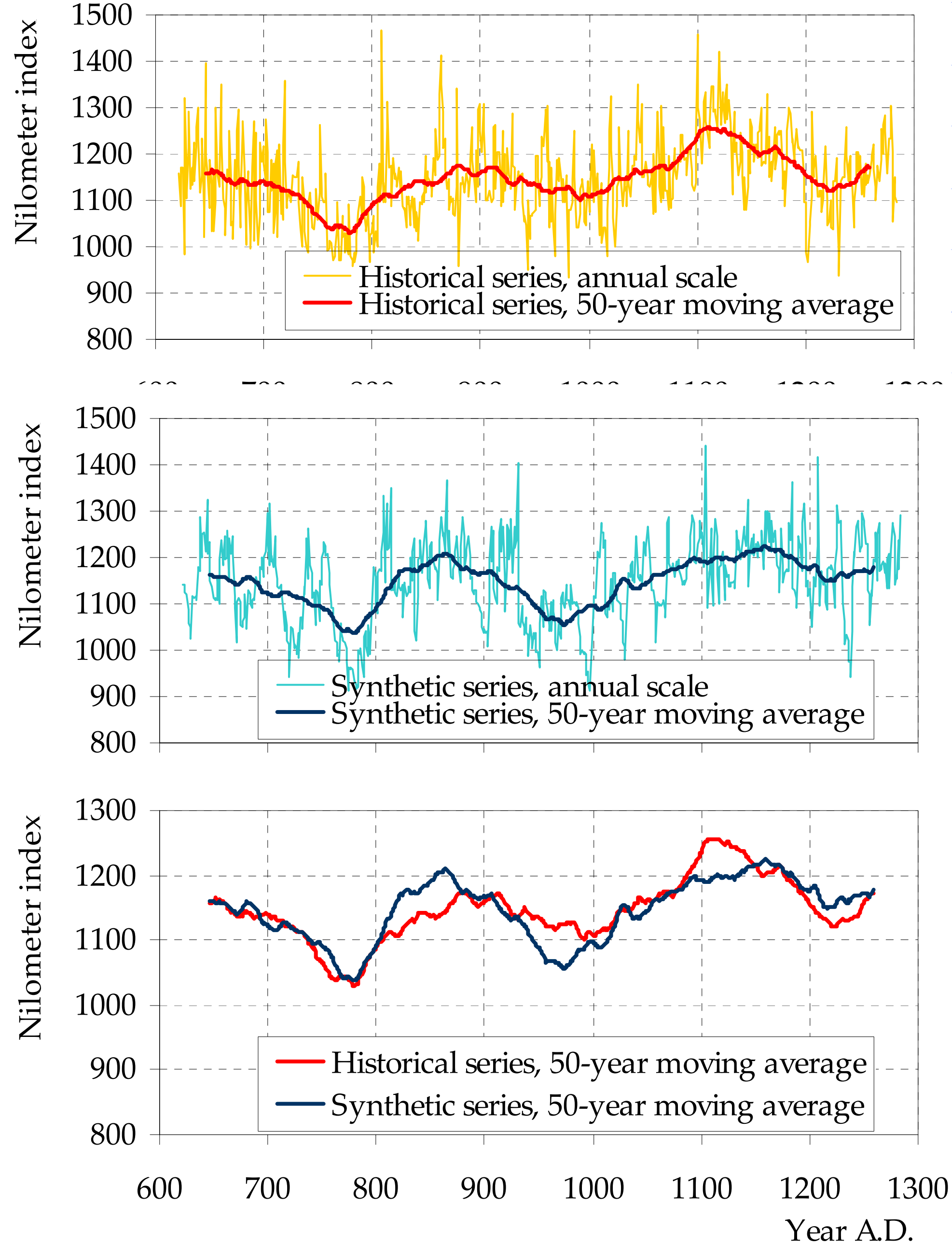
- Aim of parameter fitting to the example data sets: Generation of a synthetic series that resembles
  - downward and upward trends
  - statistical properties

  of historical series
- Criteria for parameter fitting: Large correlation with historical series
  - for time scale of 1 time step
  - for time scale of 50 time steps

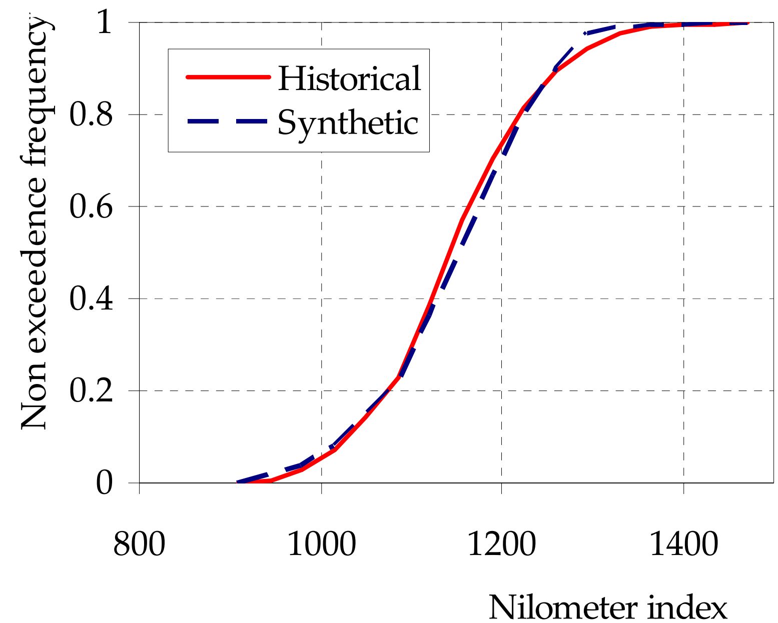
Note: In all examples, $n = 4$

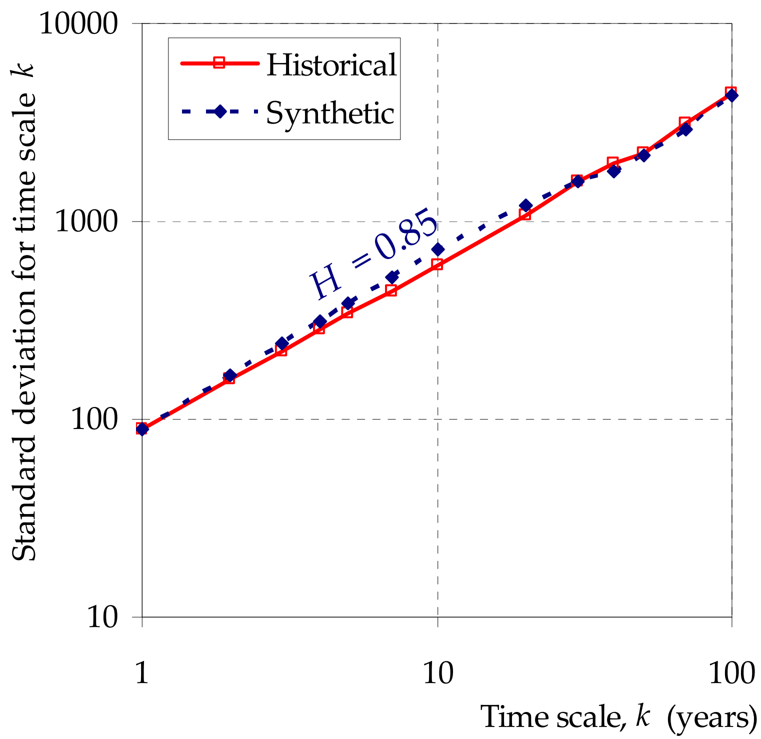
Data set	Fitted parameters					Initial values	
	κ	λ	b	c	d	z_0	α_0
Nilometer	1.871	0.477	-26871.1	28130.5	0.0013	0.030	0.335
Jones	1.765	0.317	73.3	-73.8	0.0013	0.797	0.325
Vostok	1.810	0.332	624.8	-628.6	0.0011	0.988	0.327

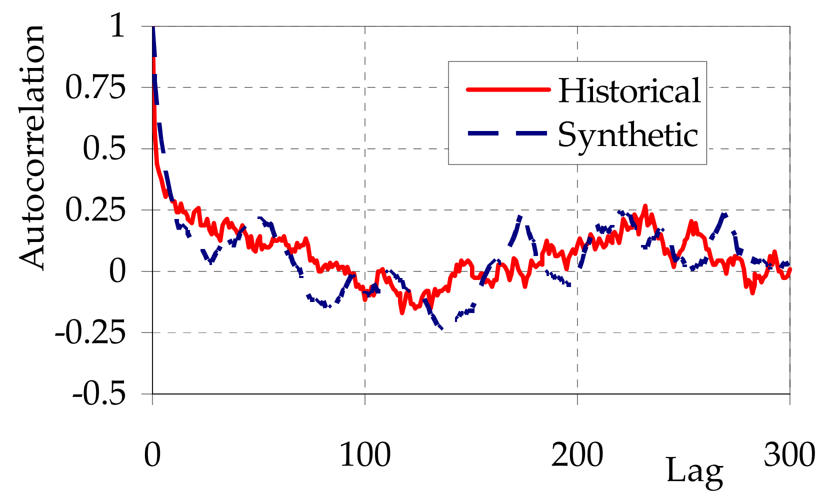
Data set: Nilometer Generated series



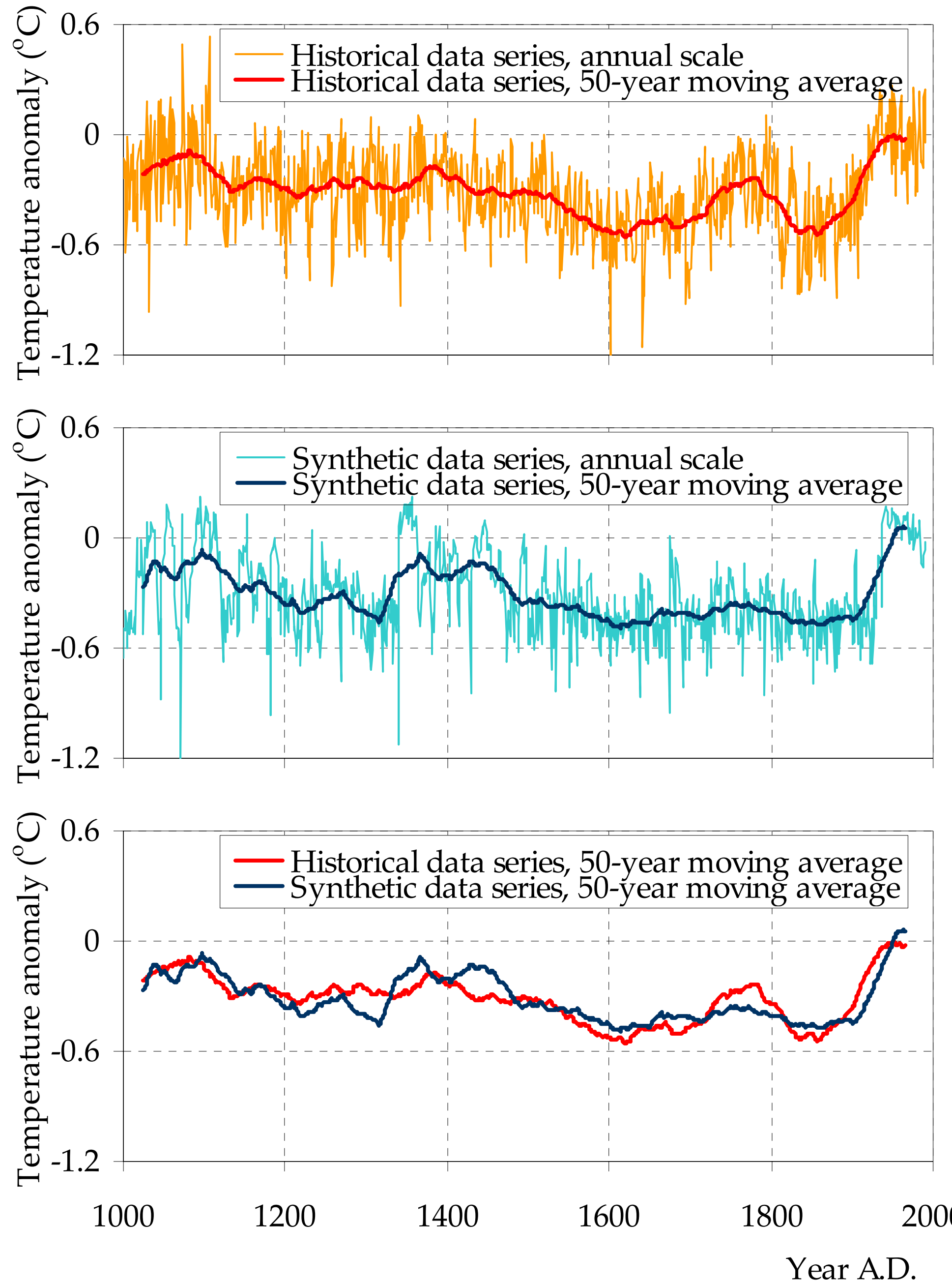
Data set: Nilometer Comparison of statistical properties between historical and generated series



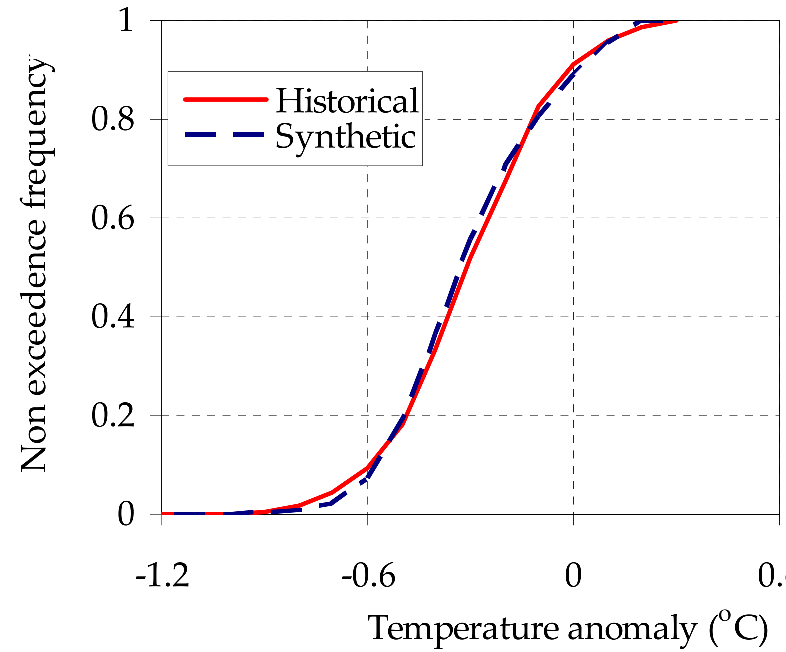


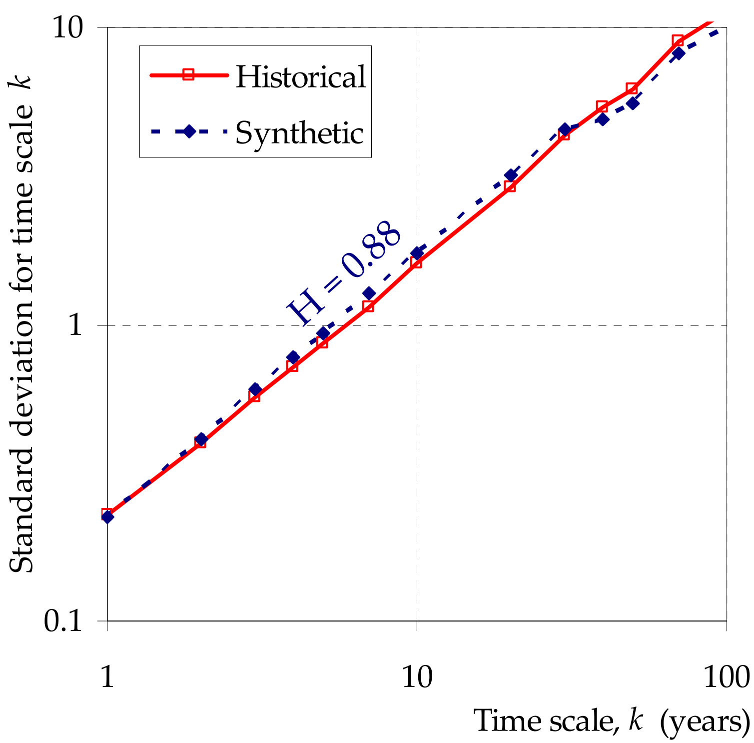


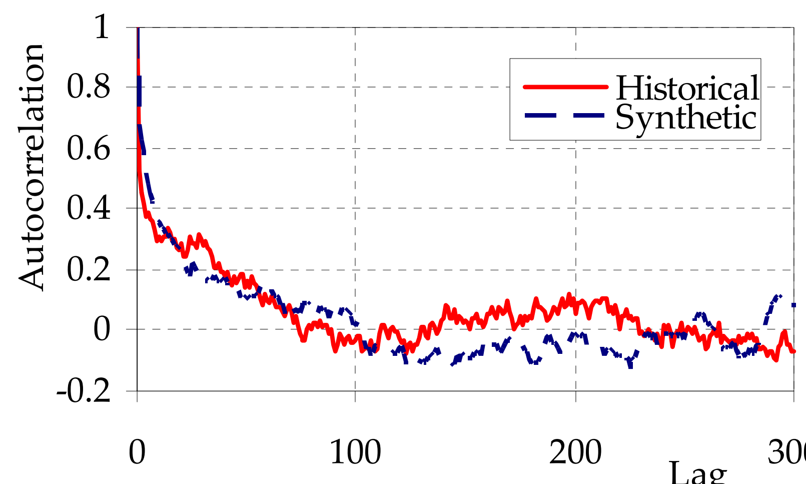
Data set: Jones Generated series



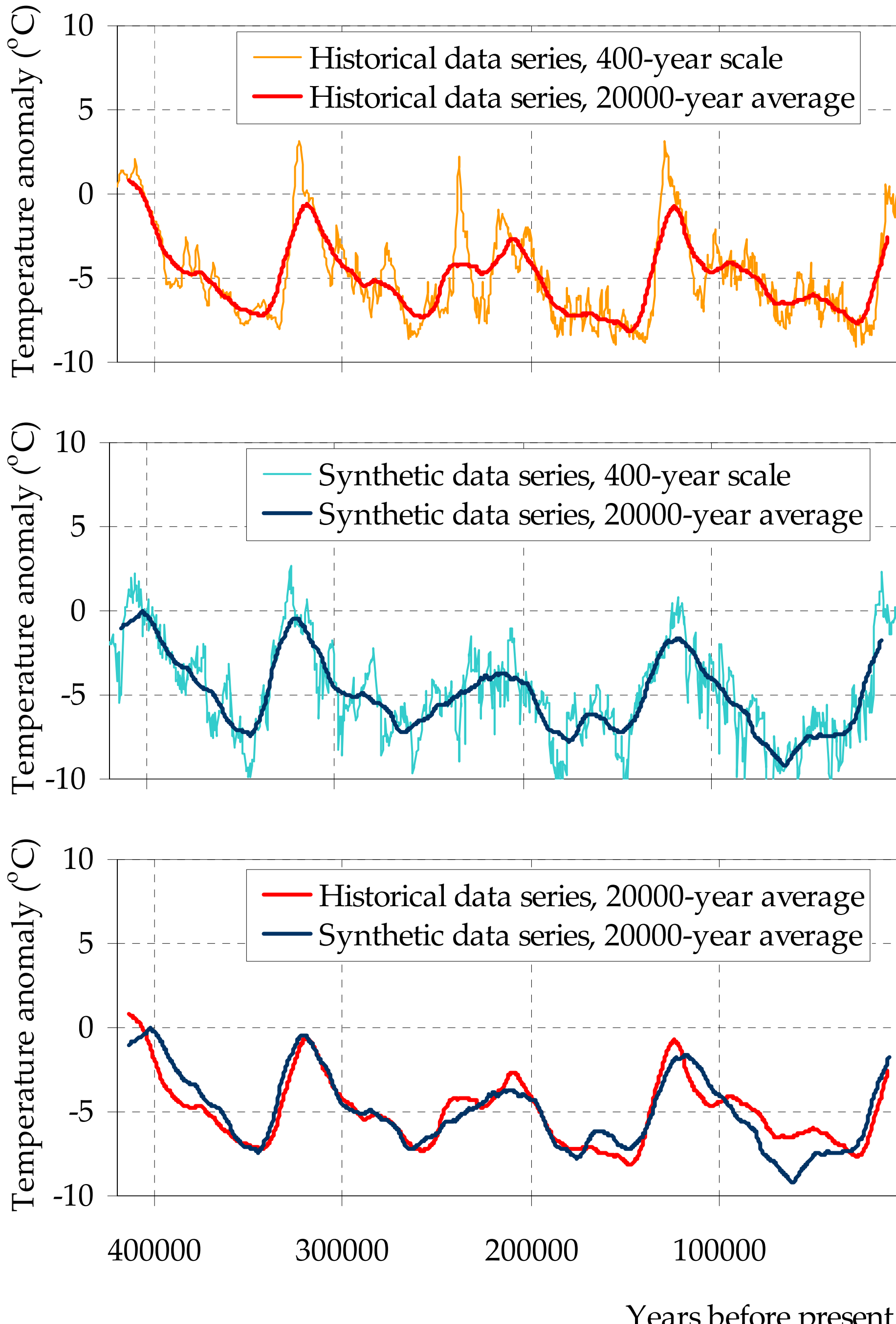
Data set: Jones Comparison of statistical properties between historical and generated series



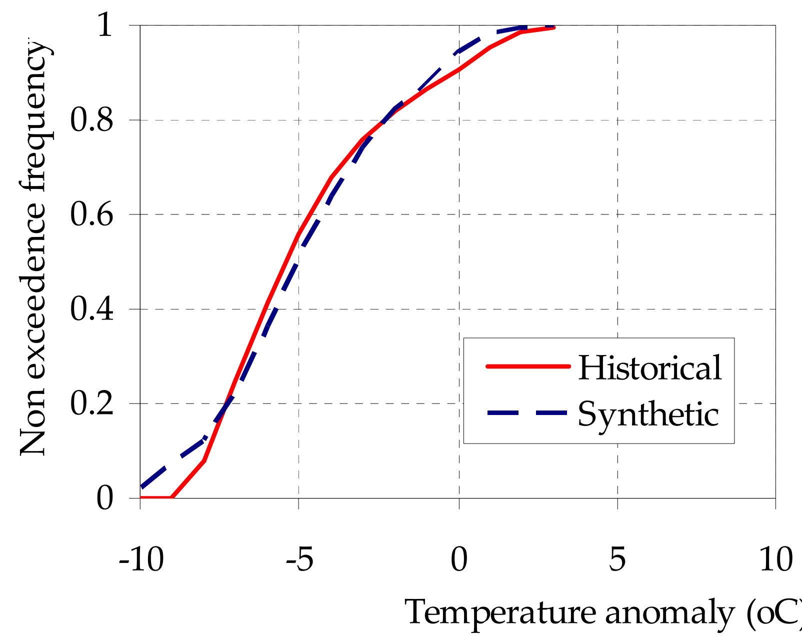


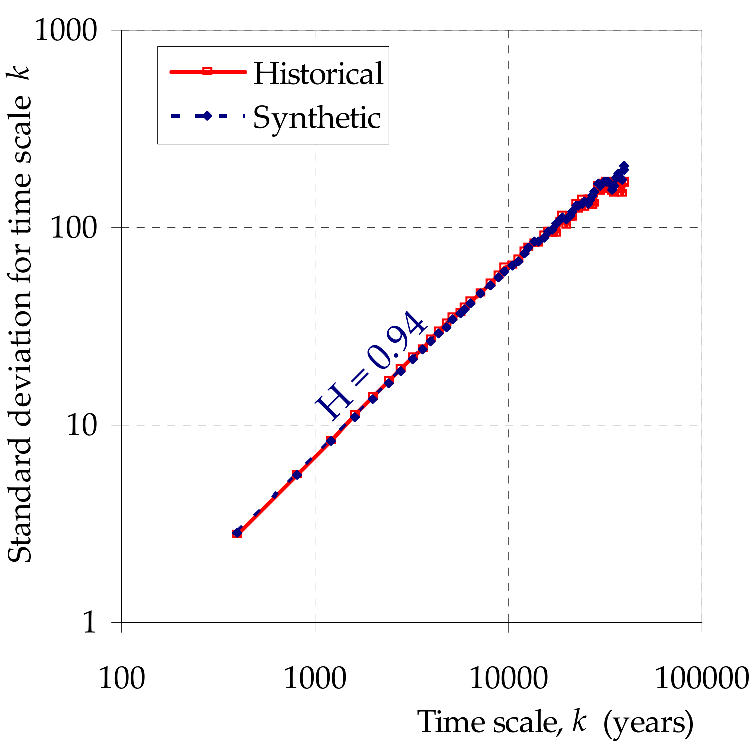


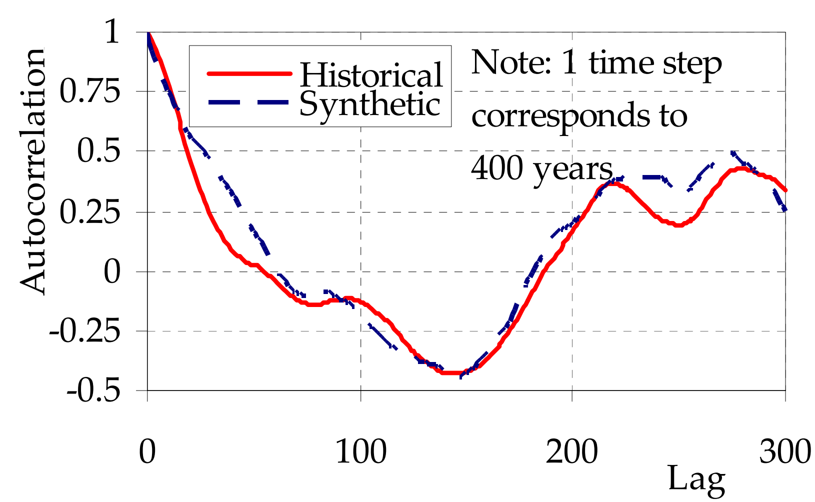
Data set: Vostok Generated series



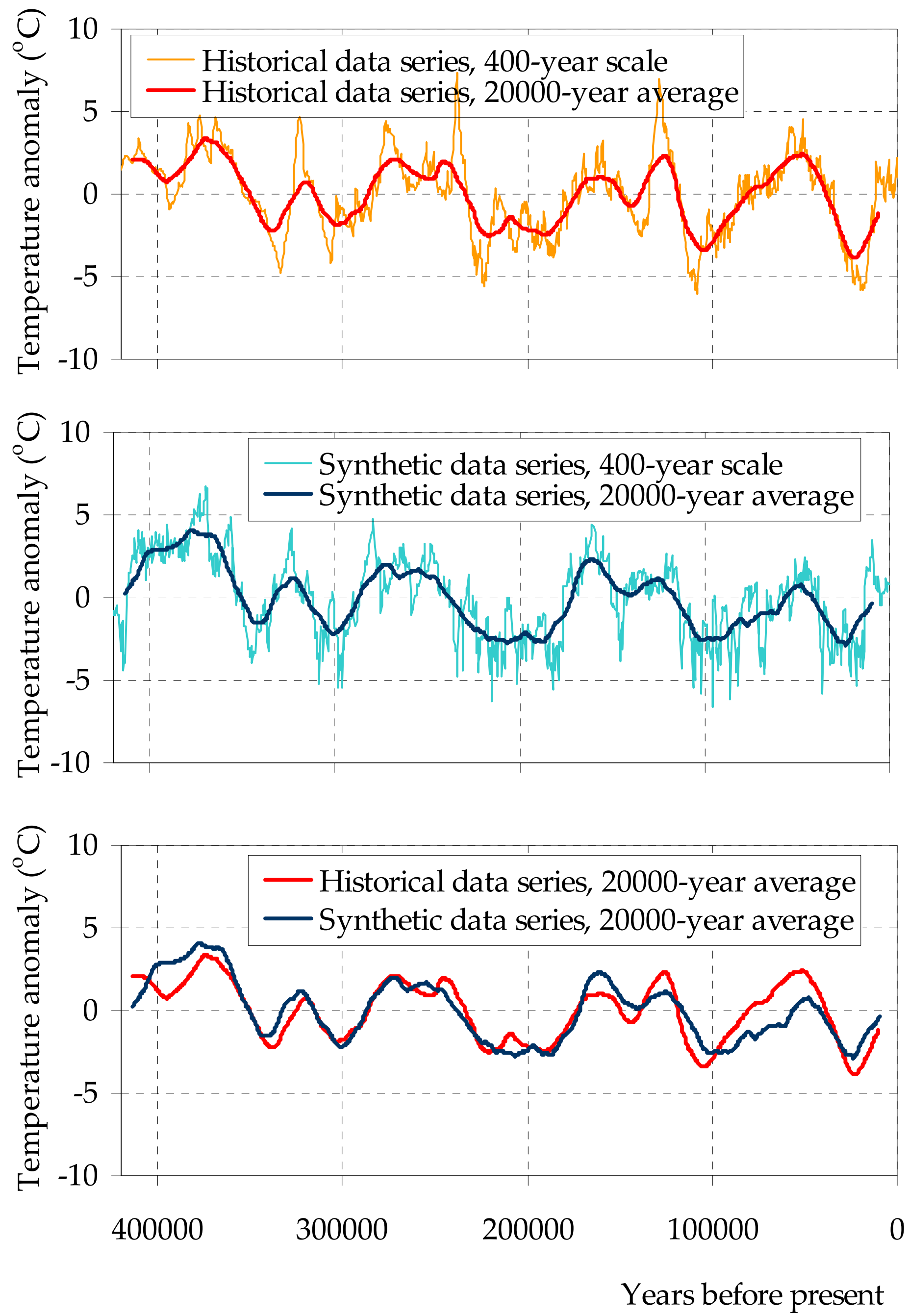
Data set: Vostok Comparison of statistical properties between historical and generated series



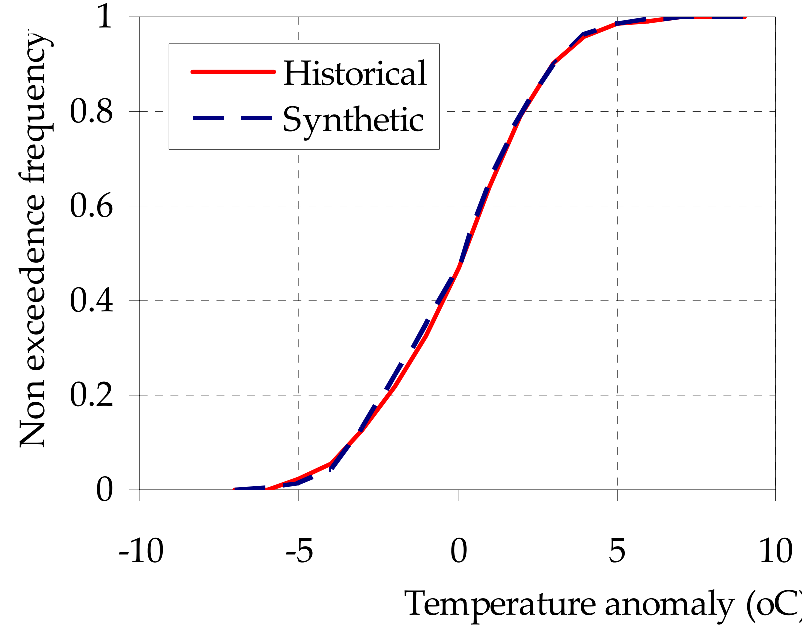


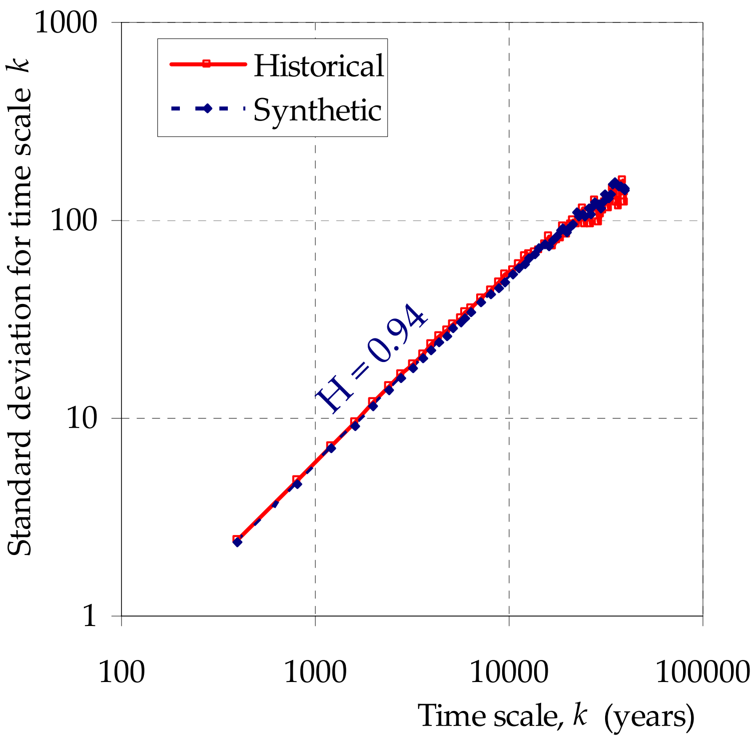


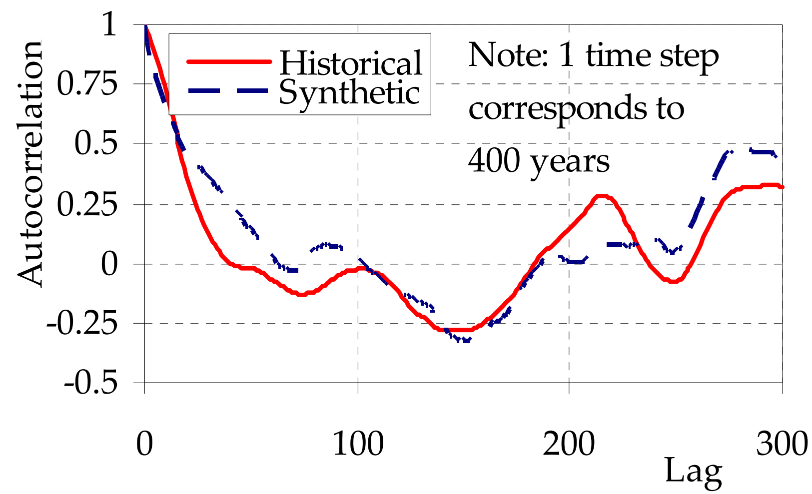
Data set: Vostok minus harmonic Generated series



Data set: Vostok minus harmonic Comparison of statistical properties between historical and generated series



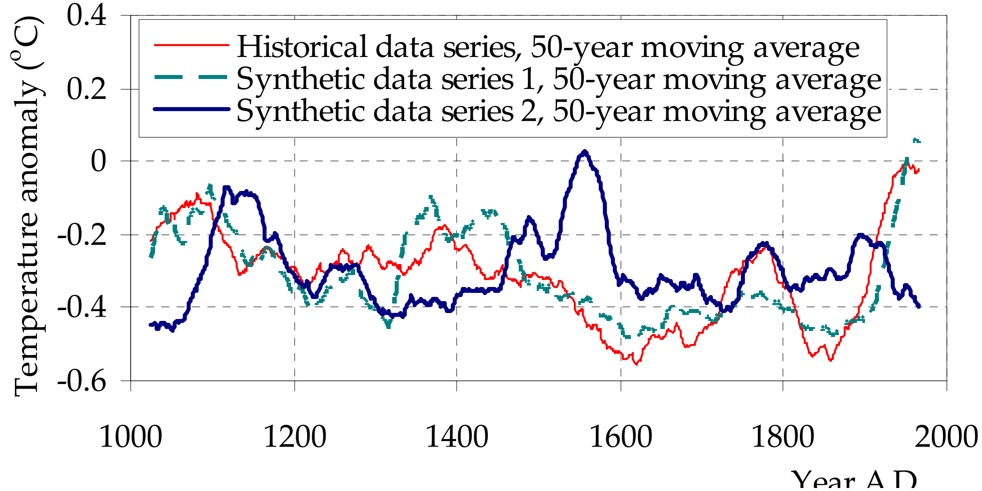


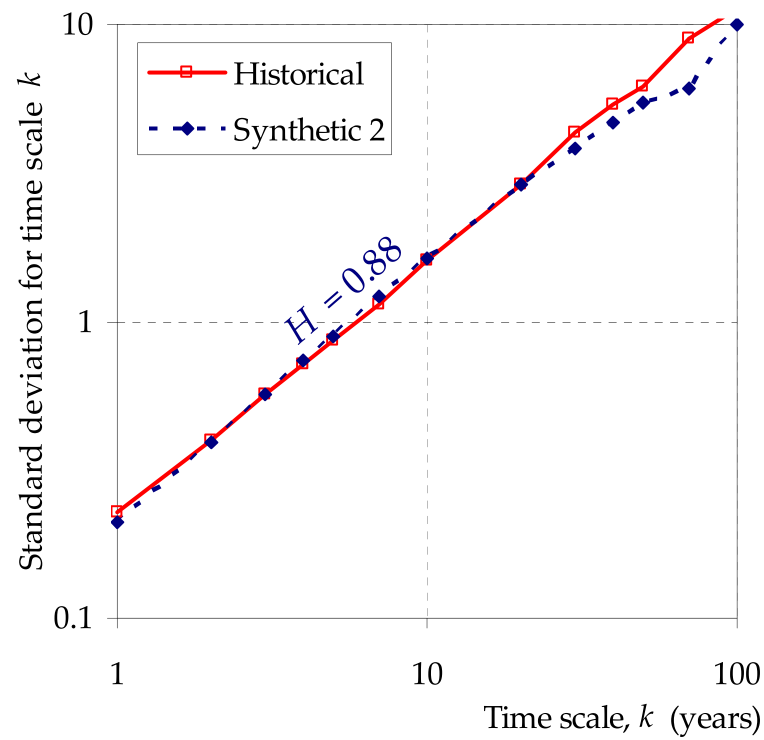


The role of initial values

Data set: Jones

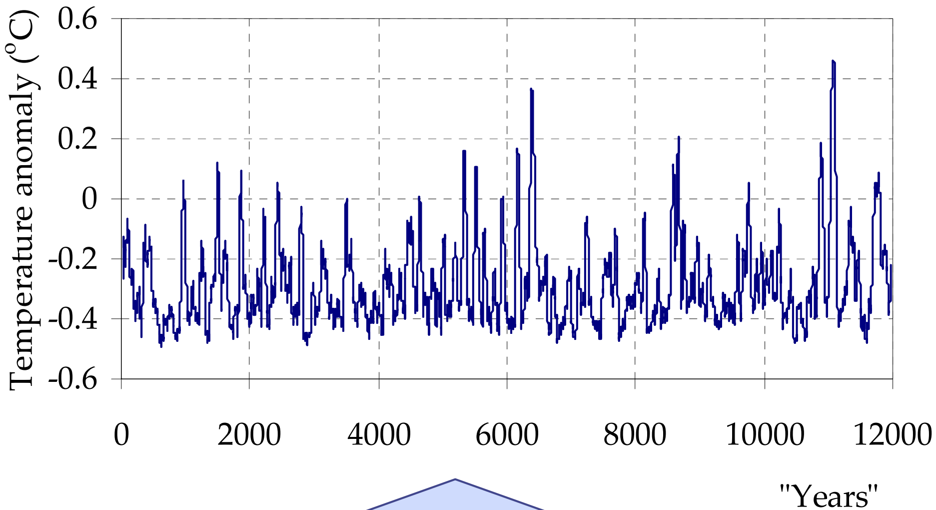
Series 2 was generated with same parameters as series 1, with same initial value z_0, but with initial value of parameter α_0 greater by 0.01%





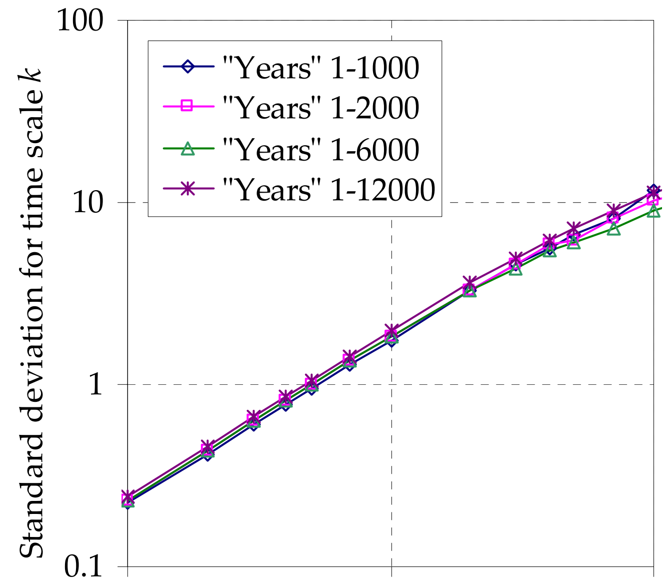
The evolution of "climate" is totally different but the statistical characteristics (especially the Hurst exponent) remain the same

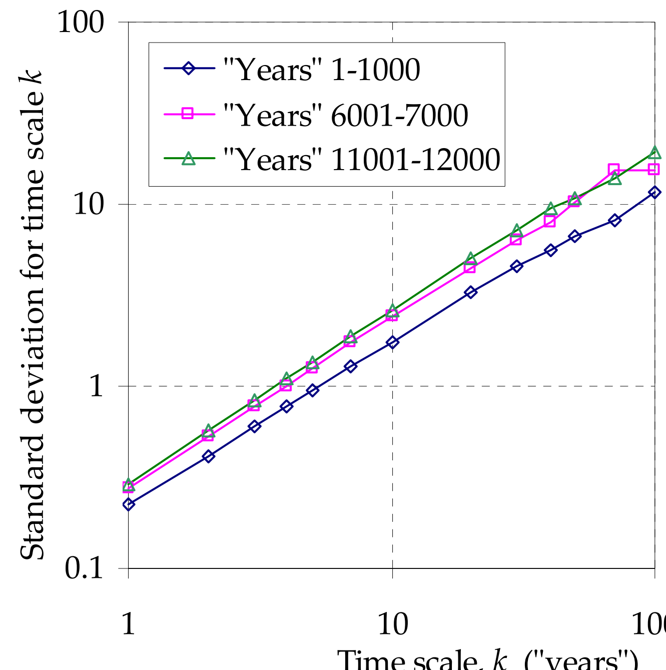
The role of series length



Data set: Jones

Series 1 extended to 12 000 "years" (plotted is the 50-"year" moving average)



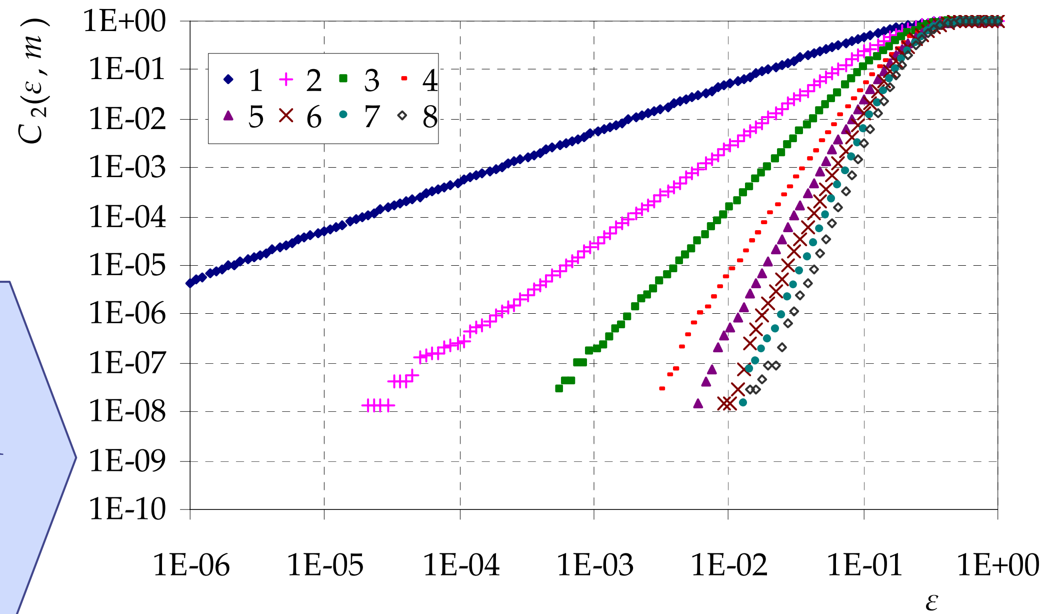


The statistical characteristics (especially the Hurst exponent) do not depend seriously on length or location within time series

Tracing of determinism in a generated series

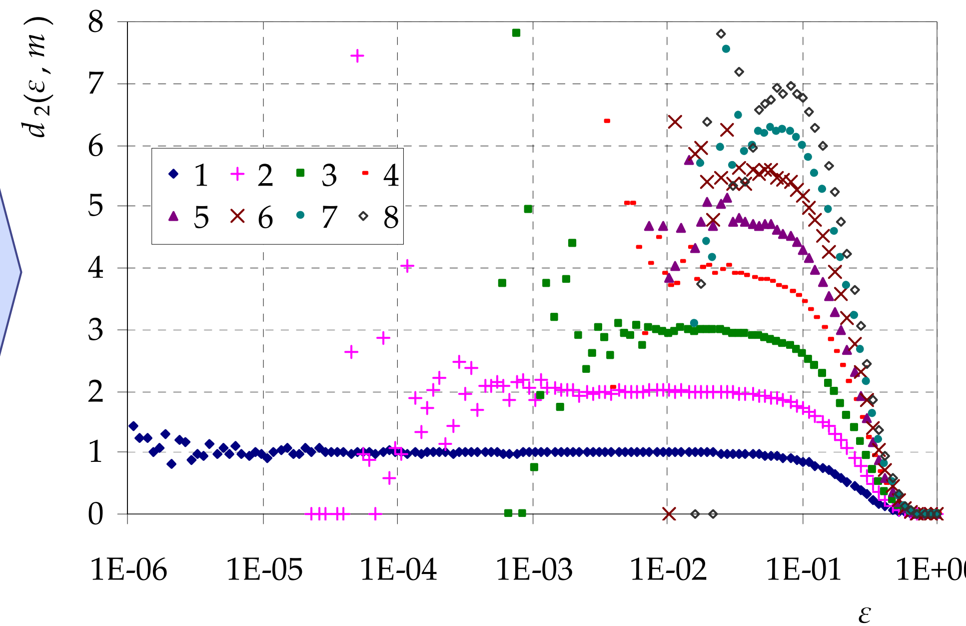
Data set: Jones

Correlation sum C_2 as a function of scale length ε and embedding dimension m for the series 1 with length N = 12 000



The slope of correlation integral $d_2(\varepsilon, m)$ increases with embedding dimension m

That is, the standard algorithm fails to capture the low dimensional determinism (D = 2) in the produced series (for N = 12 000) and deems it as a random series



Synopsis

- Long climatic time series reveal irregular changes (upward and downward fluctuations) on all time scales
- These comply with the fact that "Climate changes irregularly, for unknown reasons, on all timescales" (National Research Council, 1991, p. 21).
- The irregular changes on all scales are equivalent to a scaling behaviour of climatic series
- The scaling behaviour is quantified through a Hurst exponent greater than 0.5
- Synthetic time series with scaling behaviour are typically generated by appropriate stochastic models
- Even a simple two-dimensional deterministic toy model can reproduce the scaling behaviour of climatic processes
- The simplicity of the deterministic toy model (in comparison with stochastic models which are more complex) enables easy implementation and convenient experimentation

Synopsis (2)

- This toy model is based on the "chaotic tent map", which may represent the compound result of a positive and a negative feedback mechanism
- Application of the toy model gives traces that can resemble historical climatic time series; in particular, exhibit scaling behaviour with a Hurst exponent greater than 0.5
- Moreover, application demonstrates that large-scale synthetic "climatic" fluctuations can emerge without any specific reason and their evolution is unpredictable, even when they are generated by this simple fully deterministic model with only two degrees of freedom
- Obviously, the fact that such a simple model can generate time series that are realistic surrogates of real climatic series does not mean that the real climatic system involves that simple dynamics

Conclusion

- A simple two-dimensional deterministic dynamical system can produce series that resemble climatic series, especially their scaling behaviour with Hurst exponent > 0.5
- This simple toy model illustrates the great uncertainty and unpredictability of the climate system, showing that they can emerge even from caricature, purely deterministic, dynamics with only two degrees of freedom
- Obviously, the dynamics of the real climate system is greatly more complex than this simple toy model

This presentation is available on line at http://www.itia.ntua.gr/e/docinfo/585/

References

Andrade, J.S. Jr., I. Wainer, J. Mendes Filho andJ.E. Moreira, Self-organized criticality in the El Niño Southern Oscillation, *Physica A*, 215, 331-338, 1995.

Beran, J., *Statistics for Long-Memory Processes*, vol. 61 of Monographs on Statistics and Applied Probability. Chapman & Hall, New York, USA, 1994.

Cox, D. R., and V. Isham, Stochastic spatial-temporal models for rain, in *Stochastic Methods in Hydrology: Rain, Landforms and Floods*, edited by O.E. Barndorff-Nielsen, V.K. Gupta, V. Pérez-Abreu and E. Waymire, pp. 1-24, World Scientific, Singapore, 1998.

Freund, H., and P. Grassberger, The Red Queen's walk, *Physica A*, 190, 218-237, 1992.

Jones, P. D., Briffa, K. R., Barnett, T. P. & Tett, S. F. B., High-resolution paleoclimatic records for the last millennium: interpretation, integration and comparison with General Circulation Model control-run temperatures. *Holocene* 8(4), 455–471, 1998.

Lasota, A., and M.C. Mackey, *Chaos, Fractals and Noise, Stochastic Aspects of Dynamics*, Springer-Verlag, 1994.

National Research Council (1991) *Opportunities in the Hydrologic Sciences*, National Academy Press, Washington DC, USA.

Petit J.R., Jouzel J., Raynaud D., Barkov N.I., Barnola J.M., Basile I., Bender M., Chappellaz J., Davis J., Delaygue G., Delmotte M., Kotlyakov V.M., Legrand M., Lipenkov V., Lorius C., Pépin L., Ritz C., Saltzman E., Stievenard M., Climate and atmospheric history of the past 420,000 years from the Vostok ice core, Antarctica, *Nature*, 399, 429-436, 1999.

Wandewalle, N., and M. Ausloos, A toy model for life at the "edge of chaos", *Comput. & Graphics*, 20(6), 921-923, 1996.