A stochastic methodology for generation of seasonal time series reproducing overyear scaling behaviour

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Abstract In generating synthetic time series of hydrological processes at sub-annual scales it is important to preserve seasonal characteristics and short-term persistence. At the same time, it is equally important to preserve annual characteristics and overyear scaling behaviour. This scaling behaviour, which is equivalent to the Hurst phenomenon, has been detected in a large number of hydroclimatic series and affects seriously planning and design of hydrosystems. However, when seasonal models are used the preservation of annual characteristics and overyear scaling is a difficult task and is often ignored unless disaggregation techniques are applied, which, however, involve several difficulties (e.g. in parameter estimation) and inaccuracies. As an alternative, a new methodology is proposed that directly operates on seasonal time scale, avoiding disaggregation, and simultaneously preserves annual statistics and the scaling properties on overyear time scales. Two specific stochastic models are proposed, a simple widely used seasonal model with short memory to which long-term persistence is imposed using a linear filter, and a combination of two sub-models, a stationary one with long memory and a cyclostationary one with short memory. Both models are tested in a real world case and found to be accurate in reproducing all the desired statistical properties and virtually equivalent from an operational point of view.

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1. Introduction

Reliable planning and design of hydrosystems requires the generation of synthetic time series at sub-annual (e.g. monthly) time scales for more than one location simultaneously (e.g. simulation for design and control of reservoir systems). In this case, the statistical properties of hydrological data in sub-annual time scales are of interest. The preservation of the statistical properties at a sub-annual scale by no means implies automatic preservation of the same properties at the annual or multi-annual scale. It is now generally recognised that multi-annual scales are characterised by the long term persistence (Hurst phenomenon; Hurst, 1951 and more recently Haslett & Raftery, 1989; Bloomfield, 1992; Eltahir, 1996; Radziejewski & Kundzewicz, 1997; Montanari *et al.*, 1997; Vogel *et al.*, 1998), or, equivalently, a scaling behaviour of hydrological processes (Koutsoyiannis, 2002b; 2003). In contrast, typical sub-annual stochastic models (e.g. Multivariate Periodic Autoregressive – MPAR – models; Bras *et al.*, 1993) are short memory models that cannot preserve annual statistical properties and, moreover, cannot respect the multi-year scaling behaviour. This behaviour affects seriously planning and design of hydrosystems (Koutsoyiannis, 2004). It is evident now, that the preservation of overyear scaling behaviour of the process is equally, if not more, important to preservation of seasonal and annual statistical properties of the process.

At the same time, there exist multivariate stationary stochastic hydrological models (e.g. backward moving average – BMA – and symmetric moving average – SMA – models; Koutsoyiannis; 2000) that can reproduce both the annual marginal statistics (including skewness which may be important for hydrological processes) and long-term persistence of the process at many locations simultaneously. As long as these models are stationary, they cannot reproduce sub-annual statistical properties.

Till now, disaggregation techniques (Valencia & Schaake 1972, 1973; Mejia & Rousselle, 1976; Tao & Delleur, 1976; Hoshi & Burges, 1979; Todini 1980; Stedinger & Vogel, 1984;

Koutsoyiannis & Xanthopoulos 1990; Koutsoyiannis, 1992, 2001; Koutsoyiannis & Manetas 1996) are the only way to produce synthetic time series consistent with hydrological processes in more than one time scales simultaneously (i.e. from seasonal through annual to multi-annual time scale). These techniques involve two or more steps, where in the first step higher-level (annual) time series are generated, which are subsequently disaggregated to finer scales. However, these techniques involve several difficulties (e.g. in parameter estimation; Lane, 1982; Stedinger and Vogel, 1984), inaccuracies (e.g. in skewness preservation; Todini, 1980) and are slow procedures.

As an alternative to the disaggregation approach, two novel multivariate stochastic hydrological models are proposed, which are cyclostationary, so that they can describe periodicity emerging at sub-annual scales, and simultaneously capable of reproducing the annual statistical properties and multi-year scaling behaviour of the process. The models have been designed as parsimonious in the number of parameters as possible. The first is a simple widely used seasonal model with short memory to which long-term persistence is imposed using a linear filter as described in section 2. The second is a combination of two sub-models, a stationary one with long memory and a cyclostationary one with short memory, which is discussed in section 3. Both models are tested in a real world case in section 4 and the conclusions are drawn in section 5.

2. Multivariate periodic autoregressive model with symmetric moving average filter (MPAR-SMAF)

2.1 Description

This model is a combination of two existing models. The first model, that forms the base of the MPAR-SMAF model, is the multivariate periodic autoregressive model of order 1 (MPAR(1); Bras *et al.*, 1985, p. 118), which is used in order to reproduce the statistical properties and short-term memory of the process at sub-annual time scale (i.e. seasonal expected values, variances, skewness and lag one autocovariances among seasons). The second model used is the SMA model (Koutsoyiannis, 2000) which is applied as a filter on the synthetic time series produced by the MPAR(1) model to reproduce multi-year scaling behaviour (i.e. the Hurst phenomenon).

2.2 Model equations

Let us assume that X_i^l is the element of the cyclostationary vector stochastic process $\mathbf{X}_i := [X_i^1, X_i^2, \ldots, X_i^v]^T$ where the superscript T denotes the transpose of a matrix or a vector. The subscript i ($i = 1, 2, \ldots$) denotes time and the superscript l ($l = 1, \ldots, v$) denotes location (v is the number of locations). The cyclostationary process has period k (i.e. k is the number of seasons of the year; if we assume that the seasons are the months then $k = 12$). For a specified s ($s = 1, \ldots, k$) and location l, the stochastic process $X^l_{(i-1)k+s}$ ($i = 1, 2, \ldots$) is stationary. For convenience, the cyclostationary stochastic process X_i^l is assumed to have zero expected value ($\mu^l_i := E[X_i^l] = 0$ for any i and l) whereas the variances, $(\sigma^l_i)^2 := \mathrm{Var}[X_i^l]$, skewness and autocorrelations vary in season.

The aggregated process at the annual scale,

$$Z^l_i := \sum_{j=(i-1)k+1}^{ik} X^l_j = \sum_{s=1}^{k} X^l_{(i-1)k+s} \ , \ l = 1, \ldots, v \tag{1}$$

is obviously a stationary stochastic process. This is assumed to have the second-order properties of a Fractional Gaussian Noise process (FGN; Mandelbrot, 1977). Thus, its autocorrelation function $\rho^l_j := \mathrm{Corr}[Z^l_i, Z^l_{i-j}]$ ($j = 0, 1, \ldots$) is

$$\rho^l_j = (1 / 2) \, [(|j| + 1)^{2H_l} + (|j| - 1)^{2H_l}] - |j|^{2H_l} , \ l = 0, 1, \ldots v; \ j = 0, \pm 1, \ldots \tag{2}$$

where H_l is the Hurst coefficient for location l. In order to preserve the Hurst phenomenon for each location l, we have to reproduce ρ^l_j. Although the autocorrelation function contains infinite terms, a finite number of terms q+1 (i.e. ρ^l_0, …, ρ^l_q) suffices in practice, since terms ρ^l_j that are smaller than a small threshold value can be neglected. It is important to mention, that not only the FGN autocorrelogram but any other type of autocorrelogram may easily be reproduced following the same methodology presented in the following analysis.

It can be easily shown (Langousis & Koutsoyiannis, 2003) that the sum of two or more stationary stochastic processes with the same Hurst coefficient is a stationary stochastic process with Hurst coefficient equal to the initial one. Thus, to preserve the Hurst coefficient H_l of Z^l_i it suffices to generate sub-sequences $X^l_{(i-1)k+s}$ each of those having Hurst coefficient H_l.

The cyclostationary process X^l_i will be generated in terms of an auxiliary cyclostationary process $\mathbf{W}_j := [W^1_j, W^2_j, \ldots, W^v_j]^T$ with period k, zero expected values ($E[W^l_j] = 0$) and unit variances ($\mathrm{Var}[W^l_j] = 1$). The process $\mathbf{W}_j$ is assumed to be an MPAR(1) process described by the equation,

$$\mathbf{W}_j = \mathbf{a}_j\, \mathbf{W}_{j-1} + \mathbf{b}_j\, \mathbf{V}_j \tag{3}$$

where $\mathbf{V}_j = [V^1_j, V^2_j, \ldots, V^v_j]^T$ is the vector of v cyclostationary stochastic processes with period k, zero expected values ($E[V^l_j] = 0$), unit variances and zero correlation in time j and among locations l ($\mathrm{Cov}[V^l_i, V^k_j] = 0$, l, k = 1, 2, …, v) and $\mathbf{a}_j$, $\mathbf{b}_j$ (j = 1, …, k) are $v \times v$ periodically changing parameter matrices.

For typical values of k (e.g. k = 4 for the seasonal scale, k = 12 for the monthly scale etc.) and for specified s and l, the sub-sequence $W^l_{(j-1)k+s}$ (j = 1, 2, …) can be regarded as uncorrelated in time j. This is due to the fast decay of the MPAR(1) autocorrelogram which is the hallmark of a short memory model (see also Figure 1). Due to its short memory, the Hurst coefficient of the process $\mathbf{W}_j$ is obviously 0.5 for all locations l. However, as shown by Koutsoyiannis (2000) by filtering the generated process $\mathbf{W}_j$ with a linear Symmetric Moving Average (SMA) filter we can generate a process $\mathbf{X}_j$ that can have any desirable autocorrelogram (in our case this is as in equation (2)). The SMA filter is described by the following equation adapted for the case examined,

$$X^l_{(i-1)k+s} = \sigma^l_s \sum_{j=-q}^{q} \alpha^l_{|j|} \, W^l_{(i+j-1)k+s} \;\; , \;\; s = 1, \ldots, k; \;\; l=1, 2, \ldots, v \qquad (4)$$

where α^l_j ($j = 0, 1, \ldots, q$) are the SMA coefficients. These coefficients can be easily estimated using the inverse finite Fourier transform $s^l_\rho(\omega)$ of the autocorrelation sequence ρ^l_j ($j = 0, 1, \ldots, q$) of the stochastic process Z^l_i. $s^l_\rho(\omega)$ is given by

$$s^l_\rho(\omega) := 2 + 4 \sum_{j=1}^{q} \rho^l_j \cos(2\pi j\omega) \;\; , \;\; \omega \in [0, ½]; \;\; l = 1, 2, \ldots, v \qquad (5)$$

Koutsoyiannis (2000) has shown that the inverse finite Fourier transform $s^l_\alpha(\omega)$ of the coefficients α^l_j is related to that of the coefficients ρ^l_j by,

$$s^l_\alpha(\omega) = \sqrt{2\, s^l_\rho(\omega)} \;\; , \;\; l = 1, 2, \ldots, v \qquad (6)$$

So, the coefficients α^l_j can be directly estimated by,

$$\alpha^l_j = \int_0^{1/2} s^l_\alpha(\omega) \cos(2\pi j\omega) \mathrm{d}\omega \;\; , \;\; j = 0, 1, 2, \ldots, q; \;\; l = 1, 2, \ldots, v \qquad (7)$$

The entire modeling procedure has two steps. In the first step, we generate the process $\mathbf{W}_j$ aiming to preserve the short memory properties of the process $\mathbf{X}_j$ whereas in the second step we focus on the long memory properties. The properties of $\mathbf{W}_j$ are related to both the SMA coefficients and the lag zero and lag one seasonal auto- and cross-covariances. We denote the lag zero variance-covariance matrices as $\mathbf{c}_j := \mathrm{Cov}[\mathbf{X}_j, \mathbf{X}_j]$ and $\mathbf{d}_j := \mathrm{Cov}[\mathbf{W}_j, \mathbf{W}_j]$, and the lag one covariance matrices as $\mathbf{g}_j := \mathrm{Cov}[\mathbf{X}_j, \mathbf{X}_{j-1}]$ and $\mathbf{h}_j := \mathrm{Cov}[\mathbf{W}_j, \mathbf{W}_{j-1}]$. We also denote the elements of each of these matrices using two superscripts; for example the (l, f)th element of $\mathbf{c}_j$ is denoted as $c_j^{l,f} = \mathrm{Cov}[X^l_j, X^f_j]$. After algebraic manipulations the following formulas are obtained

$$d_j^{l,f} = \frac{c_j^{l,f}}{\sum_{r=-q}^{q} \alpha^l_{|r|}\, \alpha^f_{|r|}}, \quad l, f = 1, \dots, v; \; j = 1, \dots, k \tag{8}$$

$$h_j^{l,f} = \frac{g_j^{l,f}}{\sum_{r=-q}^{q} \alpha^l_{|r|}\, \alpha^f_{|r|}}, \quad l, f = 1, \dots, v; \; j = 1, \dots, k \tag{9}$$

from which we can estimate analytically the elements of matrices $\mathbf{d}_j$ and $\mathbf{h}_j$ given the SMA coefficients and the elements of matrices $\mathbf{c}_j$ and $\mathbf{g}_j$. The latter are estimated from historical data.

The preservation of seasonal skewness coefficients can be done very easily. The elements of the vector of skewness coefficients $\boldsymbol{\xi}_{\mathbf{w}j} := [\xi_{\mathrm{W}\,j}^{\,1}, \xi_{\mathrm{W}\,j}^{\,2}, \dots, \xi_{\mathrm{W}\,j}^{\,v}]^T$ of $\mathbf{W}_j$, can be estimated analytically from the skewness coefficients $\boldsymbol{\xi}_{\mathbf{x}j} := [\xi_{\mathrm{X}\,j}^{\,1}, \xi_{\mathrm{X}\,j}^{\,2}, \dots, \xi_{\mathrm{X}\,j}^{\,v}]^T$ of $\mathbf{X}_j$ using the equation,

$$\xi_{\mathrm{W}\,j}^{\,l} = \frac{\xi_{\mathrm{X}\,j}^{\,l}}{\sum_{r=-q}^{q} (\alpha^l_{|r|})^3}, \quad l = 1, \dots, v; \; j = 1, \dots, k \tag{10}$$

Now, the statistical properties of the vector variable $\mathbf{V}_j$, as well as the parameter matrices $\mathbf{a}_j$, $\mathbf{b}_j$ of the MPAR(1) model, can be estimated using equations,

$$\mathbf{a}_j = \mathbf{h}_j \{\mathbf{d}_{j-1}\}^{-1}, \; j = 1, \dots, k \tag{11}$$

$$\mathbf{b}_j\, (\mathbf{b}_j)^T = \mathbf{d}_j - \mathbf{a}_j\, \mathbf{d}_{j-1}\, (\mathbf{a}_j)^T, \; j = 1, \dots, k \tag{12}$$

$$\boldsymbol{\xi}_{\mathbf{v}j} := \mu_3[\mathbf{V}_j] = \left(\mathbf{b}_j^{(3)}\right)^{-1} \{ \boldsymbol{\xi}_{\mathbf{w}j} - \mu_3[\mathbf{a}_j\, \mathbf{W}_{j-1}]\}, \; j = 1, \dots, k \tag{13}$$

where $\mathbf{b}_j^{(3)}$ is the matrix whose elements are the cubes of the elements of matrix $\mathbf{b}_j$. Equation (12) yields the matrix $\mathbf{b}_j(\mathbf{b}_j)^T$ whose decomposition specifies the matrix $\mathbf{b}_j$. The decomposition has an infinite number of solutions $\mathbf{b}_j$ if $\mathbf{b}_j(\mathbf{b}_j)^T$ is positive definite and no solution if $\mathbf{b}_j(\mathbf{b}_j)^T$ is not positive definite. For a positive definite matrix $\mathbf{b}_j(\mathbf{b}_j)^T$, two well-known algorithms are commonly used which result in two different solutions $\mathbf{b}_j$ (Bras *et al.*, 1985, p. 96;

Koutsoyiannis, 1999). The first and simpler algorithm, known as triangular or Cholesky decomposition, results in a lower triangular **b**. The second, known as singular value decomposition, results in a full **b** using the eigenvalues and eigenvectors of $\mathbf{b}_j(\mathbf{b}_j)^T$. A third algorithm has been proposed by Koutsoyiannis (1999) which is based on an optimisation framework and can be applied for both positive and not positive definite matrices $\mathbf{b}_j(\mathbf{b}_j)^T$. Overall, the MPAR-SMAF model is very simple and fast to apply as all required parameters and coefficients are estimated analytically.

3. Multivariate split model

3.1 Initial equations and assumptions

The second model is based on the assumption that is possible to reproduce cyclostationarity, short-term memory and long-term persistence by splitting the stochastic process of interest into two components, a stationary stochastic process with long memory and a cyclostationary stochastic process with short memory.

As defined in section 2.2, X_i^l is a cyclostationary stochastic process with period k, where k is the number of seasons of the year. For each location l we assume that the stochastic process X_i^l can be described by the model,

$$X_i^l = e^l_i\, Y^l_i + W^l_i\,,\ \ l = 1, \dots, v \tag{14}$$

where e^l_i (l = 1, …, v) are periodically changing parameters with period k (i.e. $e^l_i = e^l_{i+\kappa k}$ for any l and $\kappa = 0, \pm1, \pm2, \dots$) and Y^l_i and W^l_i are stochastic processes independent to each other. The stochastic process Y^l_i is stationary with zero expected value ($E[Y^l_i] = 0$) and long memory (i.e. $\beta^l_p := \mathrm{Cov}[Y^l_i, Y^l_{i-p}] \neq 0$ even for large values of p). The stochastic process W^l_i is cyclostationary with period k and zero expected values ($E[W^l_i] = 0$). We assume, that W^l_i has short memory and only its lag zero and lag one autocovariances are non zero (all others are zero, i.e. $\delta^l_{i,p} := \mathrm{Cov}[W^l_i, W^l_{i-p}] = 0, p = 2, \dots$ and $l = 1, \dots, v$).

We will now array the necessary conditions that should be maintained by the model in order to preserve the statistical properties of both X_i^l and Z^l_i. The equations

$$\mathrm{Var}[X_i^l] = (e^l_i)^2 \beta^l_0 + \delta^l_{i,0}\,,\ \ l = 1, \ldots, v;\ \ i = 1, \ldots, k \tag{15}$$

$$\mu_3[X_i^l] = (e^l_i)^3 \mu_3[Y^l_i] + \mu_3[W^l_i]\,,\ \ l = 1, \ldots, v;\ \ i = 1, \ldots, k \tag{16}$$

assure the preservation of both variance and skewness at sub-annual (seasonal) scale, for each location l. The preservation of lag one autocovariances among sub-periods (seasons), is described by the equation

$$\mathrm{Cov}[X_i^l, X^l_{i-1}] = e^l_i\, e^l_{i-1} \beta^l_1 + \delta^l_{i,1}\,,\ \ l = 1, \ldots, v;\ \ i = 1, \ldots, k \tag{17}$$

The preservation of the annual variance of each location l is described by the equation

$$\mathrm{Var}[Z^l_i] = \beta^l_0 \sum_{s=1}^{k} (e^l_s)^2 + 2 \sum_{s=1}^{k-1} \sum_{j=s+1}^{k} e^l_s\, e^l_j\, \beta^l_{j-s} + \sum_{s=1}^{k} \delta^l_{s,0} + 2 \sum_{s=2}^{k} \delta^l_{s,1}\,,\ \ l = 1, \ldots, v \tag{18}$$

As described in section 2.2, in order to preserve the Hurst coefficient for each location l, we have to reproduce the autocorrelation function of the stochastic process Z^l_i. This preservation is represented by the equation,

$$\gamma^l_p = \mathrm{Cov}[Z^l_i, Z^l_{i-p}] = \sum_{j=1}^{k} \sum_{s=1}^{k} e^l_j\, e^l_s\, \beta^l_{kp-s+j} + U(1-p)\, \delta^l_{1,1}\,,\ \ l = 1, \ldots, v;\ \ p > 0 \tag{19}$$

where $U(x)$ is Heaviside's step function with $U(x) = 1$ if $x \geq 0$ and $U(x) = 0$ otherwise.

To define the model completely we need to estimate the parameters e^l_i ($l = 1, \ldots, v$ and $i = 1, \ldots, k$) of model (14) and the statistical properties of the stochastic processes Y^l_i, W^l_i. This is a complicated issue which is resolved in four steps, shown in Figure 2, each of which is described in the following four sub-sections.

3.2 Preservation of variances and short-term autocovariances

For this step, we use equations (15)-(19) but we ignore equation (16) which refers to seasonal skewness (we will refer to skewness preservation in section 3.5). Generally, we wish to preserve q+1 elements of the autocovariance function of the stochastic process Z^l_i (i.e. γ^l_0, γ^l_1, …, γ^l_q) for some large q (e.g. 1000), but in this stage we will preserve a smaller number n+1 of them ($n < q$). The preservation of the remaining elements will be discussed latter. In this case, for each location l we have to solve a nonlinear system of $2k+n+1$ equations (i.e. k equations (15), k equations (17), one equation (18) and n equations (19)) with $k(n+4)$ unknown parameters (i.e. e^l_0, e^l_1, …, e^l_k; β^l_0, β^l_1, …, $\beta^l_{k(n+1)-1}$; $\delta^l_{1,0}$, …, $\delta^l_{k,0}$; $\delta^l_{1,1}$, …, $\delta^l_{k,1}$). Obviously, the number of unknowns is greater than the number of equations and the nonlinear system has an infinite number of solutions. This must be regarded as an advantage as it allows introducing some constraints to ensure that the unknown parameters are physically and mathematically consistent. Specifically we have used the following constraints:

$$\beta^l_0 \geq \varepsilon, \delta^l_{i,0} \geq \varepsilon, \quad l = 1, \ldots, v; \quad i = 1, \ldots, k \tag{20}$$

$$\left| \frac{\delta^l_{i,1}}{\sqrt{\delta^l_{i,0}\, \delta^l_{i-1,0}}} \right| \leq 1 - \varepsilon, \quad l = 1, \ldots, v; \quad i = 1, \ldots, k \tag{21}$$

$$\varepsilon \leq \frac{\beta^l_p}{\beta^l_0} \leq 1 - \varepsilon, \quad p = 1, \ldots, k(n+1)-1; \quad l = 1, \ldots, v \tag{22}$$

where, ε is a small positive number (e.g. ε = 0.01). Constraint (20) prohibits negative variances, whereas constraints (21) and (22) prohibit correlation coefficients greater than one. The latter also prohibits negative correlations in order to achieve a smooth seasonal autocorrelogram (see Figure 15). A solution to the nonlinear system that simultaneously obeys constraints (20)-(22) can be obtained using gradient based multivariate nonlinear optimisation (e.g. conjugated gradient methods) where the constraints can be incorporated into the objective function using the method of penalties. The objective function needed for the former optimisation, as well as the expression of its derivative have been determined analytically (Langousis & Koutsoyiannis, 2003). In order to keep the number of variables involved in the optimisation procedure as low as possible, the number of the autocovariances n maintained at this stage can be set equal to one.

3.3 Preservation of long-term autocovariances

For maintaining the next q-n autocovariances of the process at the annual scale, we should estimate the remaining coefficients $\beta^l_{k(n+1)}$, …, $\beta^l_{k(q+1)-1}$ ($k(q-n)$ unknowns) so that the covariances given by equation (19) be preserved for lags $p = n+1$, …, q (q-n equations). Again, the number of unknowns is greater than the number of equations which in this case are all linear in terms of the unknowns β^l_i. In this case we adopt the generalised inversion procedure which results a tridiagonal system of linear equations (Langousis & Koutsoyiannis, 2003) that is very easily solved.

3.4 Parameters of the generating schemes

Having already estimated the basic statistical properties of the stochastic processes Y^l_i and W^l_i for each location l, we may now array the models selected to reproduce these statistical properties and determine their parameters. The stochastic process Y^l_i can be described using a SMA (Symmetric Moving Average) model (Koutsoyiannis, 2000) given by the equation

$$Y^l_i = \sum_{j=-k(q+1)+1}^{k(q+1)-1} \alpha^l_{|j|}\, V^l_{i+j}\,,\ \ l = 1, \ldots, v \qquad (23)$$

where a^l_j ($j = 0, 1, \ldots, k(q+1)-1$) are the SMA parameters of location l and V^l_i is uncorrelated in time but correlated among locations white noise with zero expected value ($E[V^l_i] = 0$) and unit variance ($\mathrm{Var}[V^l_i] = 1$). The stochastic process V^l_i may be described by the simple multivariate model,

$$\mathbf{V}_i := [V_i^1, \ldots, V_i^v]^T = \mathbf{b}\ \mathbf{H}_i \tag{24}$$

where **b** is a $v \times v$ parameter matrix (the estimation of which will be discussed in section 3.5) and $\mathbf{H}_i := [H_i^1, \ldots, H_i^v]^T$ is the vector of v stationary stochastic processes which are uncorrelated in time and among locations with zero expected values ($E[H^l_i] = 0$) and unit variances ($\mathrm{Var}[H^l_i] = 1$).

The SMA parameters a^l_j ($j = 0, 1, \ldots, k(q+1)\text{-}1$) of each location l can be estimated from the autocovariance sequence (i.e. β^l_0, β^l_1, …, $\beta^l_{k(q+1)-1}$) of the stochastic process Y^l_i. If the autocovariance matrix of Y^l_i is feasible, then the SMA parameters a^l_j of each location l can be estimated analytically using the power spectrum $s^l_\beta(\omega)$ of the stochastic process Y^l_i,

$$s^l_\beta(\omega) := 2\,\beta^l_0 + 4 \sum_{j=1}^{k(q+1)-1} \beta^l_j \cos(2\pi j\omega)\ ,\ \ \omega \in [0, ½] \tag{25}$$

$$s^l_\alpha(\omega) = \sqrt{2\, s^l_\beta(\omega)} \tag{26}$$

$$\alpha^l_j = \int_0^{1/2} s^l_\alpha(\omega) \cos(2\pi j\omega) \mathrm{d}\omega\ ,\ \ j = 0, 1, 2, \ldots, k(q+1)\text{-}1 \tag{27}$$

It is generally expected that the autocovariance sequences will be feasible so that the spectrum $s^l_\beta(\omega)$ will be positive for any ω. However, it is possible that some small negative values will emerge since no relevant constraint was imposed in earlier phases of parameter estimation (in fact such a constraint is difficult to incorporate). For this case, a simple algorithm has been developed, which uses iteration to modify the autocovariance sequences β^l_i in order to be feasible (Langousis & Koutsoyiannis 2003) (note that infinite number of β^l_i sequences exist that satisfy equations (15)-(19)).

The stochastic process W^l_i ($l = 1, \ldots, v$) can be described by the Multivariate Periodic Forward Moving Average model,

$$\mathbf{W}_i := [W_i^1, \ldots, W_i^v]^T = {}^0\mathbf{f}_i\, \mathbf{R}_i + {}^1\mathbf{f}_i\, \mathbf{R}_{i+1} + \mathbf{g}_i\, \mathbf{G}_i \tag{28}$$

where ${}^{j}\mathbf{f}_i$ (j = 0, 1), $\mathbf{g}_i$ are $v \times v$ periodically changing parameter matrices with period k (the estimation of which is going to be discussed in section 3.5) and $\mathbf{R}_i := [R_i^1, \ldots, R_i^v]^T$, $\mathbf{G}_i := [G_i^1, \ldots, G_i^v]^T$ are vectors of v cyclostationary stochastic processes with period k, which are uncorrelated in time and among locations with zero expected values and unit variances.

3.5 Skewness preservation and multivariate consistency

The additional requirements remained to be specified in this final step is the preservation of skewness of variables and the preservation of cross-correlations among different locations. These requirements will be based on the coupling of stochastic models (23) and (28). The preservation of the variance-covariance matrix of each season is described by the following equation which is a direct consequence of (14) and (28),

$$\mathrm{Cov}[\mathbf{X}_s, \mathbf{X}_s] = \mathbf{u}_s + {}^{0}\mathbf{f}_s\,({}^{0}\mathbf{f}_s)^T + {}^{1}\mathbf{f}_s\,({}^{1}\mathbf{f}_s)^T + \mathbf{g}_s\,(\mathbf{g}_s)^T \;,\; s = 1, \ldots, k \tag{29}$$

where $\mathbf{X}_s = [X_s^1, \ldots, X_s^v]^T$ and $\mathbf{u}_s$ is a periodically changing $v \times v$ parameter matrix with period k. The elements of the latter are estimated by equation,

$$u_s^{i,j} = \tilde{u}^{i,j}\left[\sum_{r=-k(q+1)+1}^{k(q+1)-1} a^i_{|r|}\, a^j_{|r|}\right] e^i_s\, e^j_s \;,\; s = 1, \ldots, k;\; i, j = 1, \ldots, v \tag{30}$$

which is a consequence of (14) and (23). Matrix $\tilde{\mathbf{u}}$ is the variance-covariance matrix of vector $\mathbf{V}_i$ described by the equation,

$$\tilde{\mathbf{u}} := \mathrm{Cov}[\mathbf{V}_i, \mathbf{V}_i] = \mathbf{b}\,\mathbf{b}^T \tag{31}$$

the diagonal elements of which must be unit by definition (i.e. $\tilde{u}^{i,i} = \mathrm{Var}[V_j^i] = 1$). The preservation of seasonal skewness is described by the equation,

$$\mu_3[\mathbf{X}_s] = \mathbf{q}_s + ({}^{0}\mathbf{f}_s)^{(3)}\,\mu_3[\mathbf{R}_s] + ({}^{1}\mathbf{f}_s)^{(3)}\,\mu_3[\mathbf{R}_{s+1}] + (\mathbf{g}_s)^{(3)}\,\mu_3[\mathbf{G}_s] \;,\; s = 1, \ldots, k \tag{32}$$

where the meaning of the superscript $^{(3)}$ is that all elements of the matrix should be cubed element by element. The vector $\mathbf{q}_s$ with dimension v has elements given by the following expression, which is a consequence of (14), (23) and (28),

$$q_s^{\,i} = \left[\sum_{j=1}^{v} (b^{i,j})^3 \, \mu_3[H^j]\right]\left[\sum_{r=-k(q+1)+1}^{k(q+1)-1} (a^i_{|r|})^3\right](e^i_s)^3\ ,\ \ i = 1, \ldots, v;\ \ s = 1, \ldots, k \tag{33}$$

Equations (29) and (32) written for s = 1, …, k, form the basis of this estimation step in which we wish to determine parameters $\mathbf{b}$, ${}^0\mathbf{f}_s$, ${}^1\mathbf{f}_s$ and $\mathbf{g}_s$. Each of them has v^2 unknowns so the total number of unknown matrix elements is $(3k+1)v^2$. Simultaneously, we wish to determine the vectors of skewness coefficients $\mu_3[\mathbf{H}]$, $\mu_3[\mathbf{R}_s]$, $\mu_3[\mathbf{G}_s]$ (a total of $(2k+1)v$ unknowns). Knowing that the matrix Cov$[\mathbf{X}_s, \mathbf{X}_s]$ in (29) is symmetric, the number of independent equations, which are non-linear, is $kv(v+3)/2$ (i.e. smaller than the number of unknowns). Thus, again we have an optimisation problem which can be resolved using gradient based nonlinear optimisation.

A number of constraints need to be incorporated into this optimisation. First we need a constraint that all the diagonal elements of matrix $\tilde{\mathbf{u}} = \mathbf{b}\,\mathbf{b}^T$ are unity, i.e.

$$\tilde{u}^{\,i,\,i} = 1\ \ ,\ \ i = 1, \ldots, v \tag{34}$$

Next we need a constraint for the preservation of lag one autocovariances among seasons of the same location, i.e.

$$\begin{aligned}\mathrm{diag}\{\mathrm{Cov}[X^1_s, X^1_{s-1}], \ldots, \mathrm{Cov}[X^v_s, X^v_{s-1}]\} &= \mathrm{diag}\{\omega_s^{\,1}, \ldots, \omega_s^{\,v}\} + \\ &+ \mathrm{diag}\{{}^0\mathbf{f}^1_s\,({}^1\mathbf{f}^1_{s-1})^T, \ldots, {}^0\mathbf{f}^v_s\,({}^1\mathbf{f}^v_{s-1})^T\}\ \ ,\ \ s = 1, \ldots, k\end{aligned} \tag{35}$$

where ${}^0\mathbf{f}^{\,j}_s$ is the j^{th} line of matrix ${}^0\mathbf{f}_s$, and

$$\omega_s^{\,l} = \tilde{u}^{\,l,\,l}\left(\sum_{r=-k(q+1)+1}^{k(q+1)-2} a^l_{|r|}\, a^l_{|r+1|}\right) e^l_s\, e^l_{s-1}\ ,\ \ l = 1, \ldots, v;\ \ s = 1, \ldots, k \tag{36}$$

A final set of constraints should be incorporated in order to avoid high values of skewness coefficients of the auxiliary processes H^l, R^l_i and G^l_i, knowing that very high coefficients of skewness cannot be generated in finite length series (Todini, 1980). In this case, we set an upper limit ξ_{max} for the absolute value of skewness coefficients, i.e.

$$|\mu_3[H^l]| \leq \xi_{max}\ ,\ \ l = 1, \ldots, v \tag{37}$$

$$|\mu_3[R^l_i]|,\ |\mu_3[G^l_i]| \leq \xi_{max}\ ,\ \ i = 1, \ldots, k;\ \ l = 1, \ldots, v \tag{38}$$

All constraints can be incorporated into the objective function using, once more, the method of penalties. The objective function needed for the optimisation, as well as the expression of its derivative have been determined analytically (Langousis & Koutsoyiannis, 2003).

4. Applications

4.1 Reproduction of the statistical properties of hydrological data

In order to check the efficiency of MPAR-SMAF and Split models, we applied them for the reproduction of the statistical properties of two cross-correlated real-world monthly hydrological time series. The first time series is the longest available monthly discharge record in Greece, the Boeoticos Kephisos river discharge series, and the second time series is the monthly rainfall time series at the Aliartos rain gauge, which is the oldest station existing in the Boeoticos Kephisos basin (Koutsoyiannis *et al,* 2002).

The parameters of both models were estimated based on the historical statistics (Langousis, 2003). The long-term persistence (overyear scaling behaviour) was described assuming that the annual discharge and rainfall time-series follow the FGN (Fractional Gaussian Noise) autocorrelogram that is determined totally by the Hurst coefficient (H_l) of the annual time series (equation (2)). One set of series for all locations was then generated using each model with length of 5 000 years (i.e. 60 000 months). The statistical properties of the generated series were subsequently compared to the historical statistics.

Referring to the monthly time scale, both MPAR-SMAF and Split models preserve the seasonal expected values (Figures 3, 4), standard deviations (Figures 5, 6), skewness coefficients (Figures 7, 8) and lag one autocorrelation coefficients (Figures 9, 10) of both discharge and rainfall time series. Also, both models preserve the seasonal lag zero cross-correlation coefficients between the discharge and rainfall time series (Figure 11).

Referring to the annual time scale, we observe that both models reproduce the annual expected values of the discharge and rainfall time series (Figure 12). Although Split model reproduces the annual standard deviation of both time series (as long as this is implied by equation (18)), MPAR-SMAF model yields only a good approximation of it (Figure 12). This is explained by the structure of the MPAR-SMAF model that uses an MPAR(1) model to generate the initial time series on which SMA filter is then applied. It is evident, that series generated by an MPAR(1) model preserve only lag one autocovariances among months (that leads to a good approximation of the annual standard deviation) and not the whole autocovariance sequence (12 autocovariances for each month) needed for the accurate reproduction of the annual standard deviation. Further, as we observe in Figures 13 and 14 both models are capable of reproducing the overyear scaling behaviour of the historical time series.

Although it is not of direct interest (and not explicitly preserved by the models), Figure 15 depicts the monthly autocorrelogram (autocorrelogram of one month of a certain location with previous months of the same location) for the month of October. This provides information about the functioning structure of the two models developed. Comparing the monthly autocorrelogram of the MPAR(1) model with that of the MPAR-SMAF model, we observe that the autocorrelogram of the former tends quickly to zero due to the short memory of the model. However, the SMA filter that is applied on the series produced by the MPAR(1) model performs a periodical shift (period 12 months). In the aggregated (annual) scale, this shift is capable of reproducing the long-term persistence of the historical time series. In contrast, the Split model yields a smooth structure of the monthly autocorrelogram.

4.2 Operational comparison of the results of the two models

The analysis of the previous sub-section showed that both models can preserve several important statistical characteristics of hydrological processes. We have seen that there are some differences between the two models. Particularly, the MPAR-SMAF model approximates the variance at the annual scale whereas the Split model preserves it exactly. Moreover, even though both models yield the same annual autocorrelogram, the former model results in monthly autocorrelogram which may have an unrealistic periodic form, whereas the latter model yields a smooth realistic monthly autocorrelogram. On the other hand the former model is extremely simpler than the latter so a question arises, whether these differences are so important as to prefer the latter model over the former. To give a pragmatic answer to this question, one needs to test the model results in operational problems of stochastic hydrology. In this respect, the most common and effective test is related to reservoir design and operation. Specifically, the reservoir storage-yield-reliability relationship is regarded to be one of the most common tools to compare different models in an operational manner. Here we use this approach to compare the two models and also to intercompare them to the standard MPAR(1) model.

Let c the reservoir storage capacity of a hypothetical reservoir and δ the water demand from this reservoir on an annual basis assumed to be constant in all years. According to a general definition (Chow et al., 1988, p. 434) the reliability of a system is the probability that the “loading” will not exceed the “capacity” at a specified period of time. Applied to a reservoir on an annual basis, this definition implies that the reliability of the reservoir is the probability that the reservoir will satisfy the required demand during the whole year, i.e.

$$a = P[R_t = \delta] \tag{39}$$

where R_t is the reservoir release given by,

$$R_t = \min(S_{t-1} + X_t, \delta) \tag{40}$$

X_t is the reservoir inflow as generated by the model used and S_t is the reservoir storage given by,

$$S_t = \max[0, \min(S_{t-1} + X_t - \delta, c)] \tag{41}$$

The reliability is related to the recurrence interval of emptiness T by the equation

$$T = 1 / (1 - a) \tag{42}$$

Given the reservoir storage c and the demand δ, as well as the inflows X_t a simulation can be performed based on equations (40) and (41) and by counting the number of successes the reliability α and the recurrence interval T are estimated (equations (39) and (42)). Repeating this procedure several times with different values of c and δ, the relationship among c, δ, T can be determined. Usually, the quantities c and δ are standardised by the mean μ or the standard deviation σ of the annual inflows.

This procedure was applied with three synthetic series with length 20 000 years (to assure accuracy of calculations) generated by the MPAR(1), Split and MPAR-SMAF models for the Boeotikos Kephisos river inflows. The results are shown in Figure 16 and suggest that models Split and MPAR-SMAF are practically equivalent to each other and both differ tremendously from the MPAR(1) model especially for large standardised reservoir capacities or large standardised demands. These tremendous differences from the MPAR(1) model emphasise the major importance of preserving the overyear scaling behavior of hydrological processes and concur with the already known result (e.g. Koutsoyiannis, 2004) that the lack of preservation of the long term persistence causes significant underestimation of the reservoir characteristics. For this reason, a model without ability to preserve long term persistence should be not used for generation of hydrological inputs. On the other hand, the close results of the two models developed in this paper lead us to conclude that even the simple MPAR-SMAF is operationally as good as the more complicated Split model.

5. Conclusions

The reproduction of the scaling behaviour or equivalently the long term persistence of hydrological processes turns out to be essential in modelling and operation of hydrosystems. This reproduction may be regarded as a routine task when the modelling time scale is annual. However, the annual scale is not appropriate for simulation of most hydrosystems. Until now, only disaggregation approaches which combine different stochastic models that use different time scales (annual and seasonal) could give a solution to the problem of simultaneous reproduction of over year scaling and seasonal statistical characteristics. In this paper an alternative approach is presented, according to which one single model can directly generate synthetic time series that are consistent with historical data in several time scales, from seasonal to multiyear simultaneously, avoiding the use of disaggregation, which involves several difficulties (e.g. in parameter estimation) and inaccuracies. Following this approach, two specific stochastic models have been studied. The first one (MPAR-SMAF) a simple model based on the widely used MPAR(1) model combined with a Symmetric Moving Average filter. The latter takes the outputs of the MPAR(1) model and filters them adding the required long term persistence. The second model (Split model) reproduces seasonal characteristics and short- and long-term persistence by combining two sub-models, a stationary one with long memory and a cyclostationary one with short memory. Split model is more complete with respect to the reproduced statistical properties but simultaneously is far more complicated, especially in parameter estimation which requires several steps of nonlinear multivariate optimisation. MPAR-SMAF does not require any optimisation and its parameters are determined by analytical equations. Both models have been tested in a real world case and found to be accurate in reproducing all the desired statistical properties and virtually equivalent to each other from an operational point of view.

References

Bloomfield, P. (1992) Trends in global temperature. *Climatic Change* **21**, 1-16.

Bras, R.L. & Rodriguez-Iturbe, I. (1985) *Random Functions in Hydrology*, Addison-Wesley, U.S.A.

Chow, V.T., Maidement, D.R. and Mays. L.W. (1988) *Applied Hydrology*, McGraw-Hill, New York, U.S.A.

Eltahir, E.A.B. (1996) El Niño and the natural variability in the flow of the Nile River. *Wat. Resour. Res.*, **32**(1), 131-137.

Haslett, J. & Raftery, A.E. (1989) Space time modelling with long - memory dependence: assessing Ireland's wind power resource. *Appl. Stat.* **38**(1), 1-50.

Hoshi, K. & Burges, S.J. (1979) Disaggregation of streamflow volumes. *Journal of the Hydraulics Division, Proceedings ASCE*, **105** (HY1), 27-41.

Hurst, H.E. (1951) Long term storage capacities of reservoirs. *Trans ASCE* **116**, 776-808.

Koutsoyiannis, D. (1992) A nonlinear disaggregation method with a reduced parameter set for simulation of hydrologic series. *Wat. Resour. Res.*, **28**(12), 3175-3191.

Koutsoyiannis, D. (1999) Optimal decomposition of covariance matrices for multivariate stochastic models in hydrology. *Wat. Resour. Res.*, **35**(4), 1219-1229.

Koutsoyiannis, D. (2000) A generalized mathematical framework for stochastic simulation and forecast of hydrologic time series. *Wat. Resour. Res.*, **36**(6),1519-1534.

Koutsoyiannis, D. (2001) Coupling stochastic models of different time scales, *Wat. Resour. Res.*, **37**(2), 379-392.

Koutsoyiannis, D. (2002) The Hurst phenomenon and fractional Gaussian noise made easy. *Hydrological Sciences Journal*, **47**(4), 573-595.

Koutsoyiannis, D. (2003), Climate change, the Hurst phenomenon, and hydrological statistics. *Hydrological Sciences Journal*, **48**(1), 3-24.

Koutsoyiannis, D. (2004) Reliability concepts in reservoir design, *The Encyclopedia of Water*, edited by J. H. Lehr, Wiley, New York, U.S.A.

Koutsoyiannis, D., & Xanthopoulos, Th. (1990) A dynamic model for short-scale rainfall disaggregation *Hydrological Sciences Journal*, **35**(3), 303-322.

Koutsoyiannis, D. & Manetas, A. (1996) Simple disaggregation by accurate adjusting procedures, *Wat. Resour. Res.*, **32**(7), 2105-2117.

Koutsoyiannis, D., A. Efstratiadis, G. Karavokiros, A. Koukouvinos, N. Mamassis, I. Nalbantis, E. Rozos, Ch. Karopoulos, A. Nassikas, E. Nestoridou, and A. Nikolopoulos, Master plan of the Athens water resource system — Year 2002–2003, *Modernisation of the supervision and management of the water resource system of Athens*, 215 pages, Department of Water Resources, Hydraulic and Maritime Engineering - National Technical University of Athens, Athens, December 2002.

Lane, W.L. (1982) Corrected parameter estimates for disaggregation schemes, in *Statistical Analysis of Rainfall and Runoff*, edited by V.P. Singh, Water Resour. Publ., Highlands Ranch, Colo.

Langousis, A. (2003), *Developement of cyclostationary stochastic hydrological models preserving short-term memory and long-term persistence*, Diploma thesis, 327 pages, Department of Water Resources, Hydraulic and Maritime Engineering - National Technical University of Athens.

Langousis, A., & Koutsoyiannis, D. (2003), Mathematical derivations for seasonal models reproducing overyear scaling behaviour, Internal report, Department of Water

Resources, Hydraulic and Maritime Engineering - National Technical University of Athens (http://www.itia.ntua.gr/getfile/599/2/2003StochMethodsSeasonalScaleApp.pdf).

Mandelbrot, B.B. (1977) *The Fractal Geometry of Nature*, 248. Freeman, New York, U.S.A.

Mejia, J.M. & Rousselle, J. (1976) Disaggregation models in Hydrology revisited. *Wat. Resour. Res.*, **12**(2), 185-186.

Montanari, A., Rosso, R., Taqqu, M.S. (1997) Fractionally differenced ARIMA models applied to hydrologic time series. *Wat. Resour. Res.*, **33**(5), 1035-1044.

Radziejewski, M. & Kundzewicz, Z.W. (1997) Fractal analysis of flow of the river Warta. *J. Hydrol.* **200**, 280-294.

Stedinger, J.R. & Vogel, R.M. (1984) Disaggregation procedures for generating serially correlated flow vectors. *Wat. Resour. Res.*, **20**(1), 47-56.

Tao, P.C. & Delleur, J.W. (1976) Multistation, multiyear synthesis of hydrologic time series by disaggregation. *Wat. Resour. Res.*, **12**(6), 1303-1312.

Todini, E. (1980). The preservation of skewness in linear disaggregation schemes. *J. Hydrol.* **47**, 199-214.

Valencia, D. & Schaake, J.C. (1972) A disaggregation model for time series analysis and synthesis, Report no. 149, Ralph M. Parsons Laboratory of Water Resources and Hydrodynamics, Massachusetts Institute of Technology, Cambridge, Mass.

Valencia, D. & Schaake, J.C. (1973) Disaggregation processes in Stochastic Hydrology. *Wat. Resour. Res.*, **9**(3), 211-219.

Vogel, R.M., Tsai, Y., Limbrunner, J.F. (1998) The regional persistence and variability of annual steamflow in the United States. *Wat. Resour. Res.*, **34**(12), 3445-3459

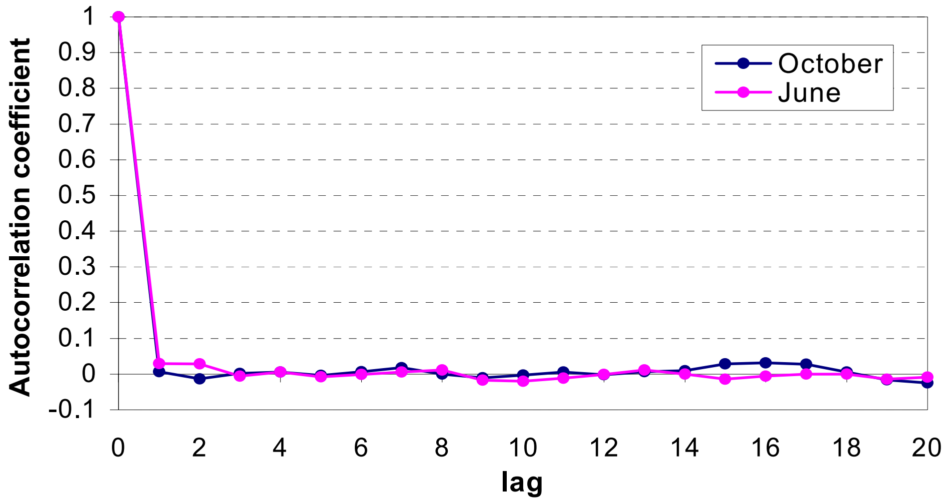

Figure 1 Empirical autocorrelation coefficients of the synthetic monthly series of the months of October and June, generated by an MPAR(1) model (the lag is expressed in years).

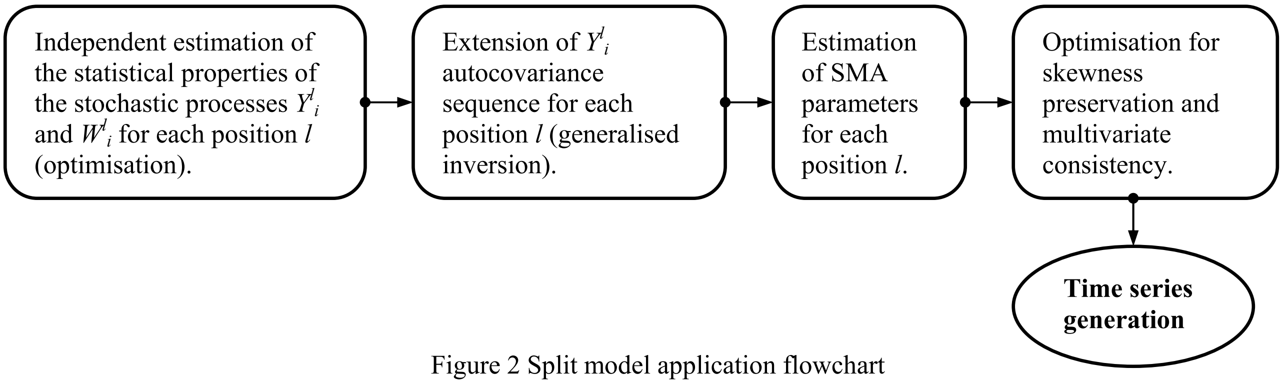

Figure 2 Split model application flowchart

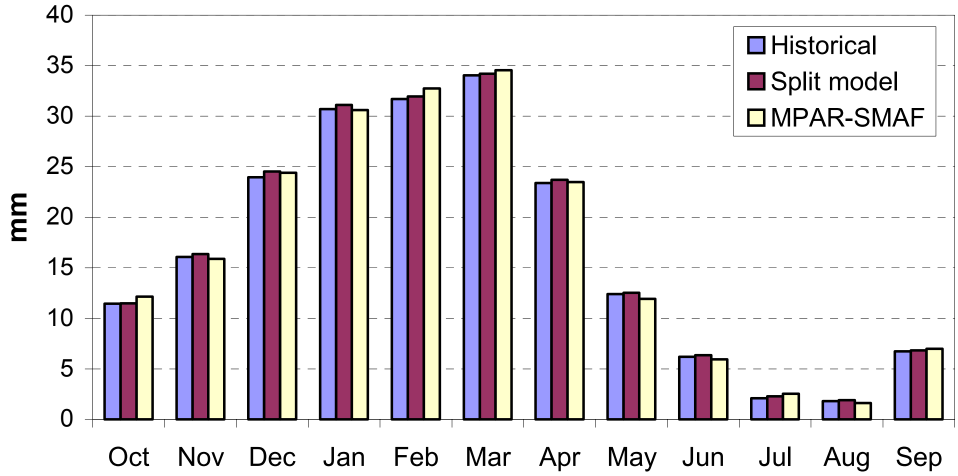

Figure 3 Monthly expected values of the discharge time series

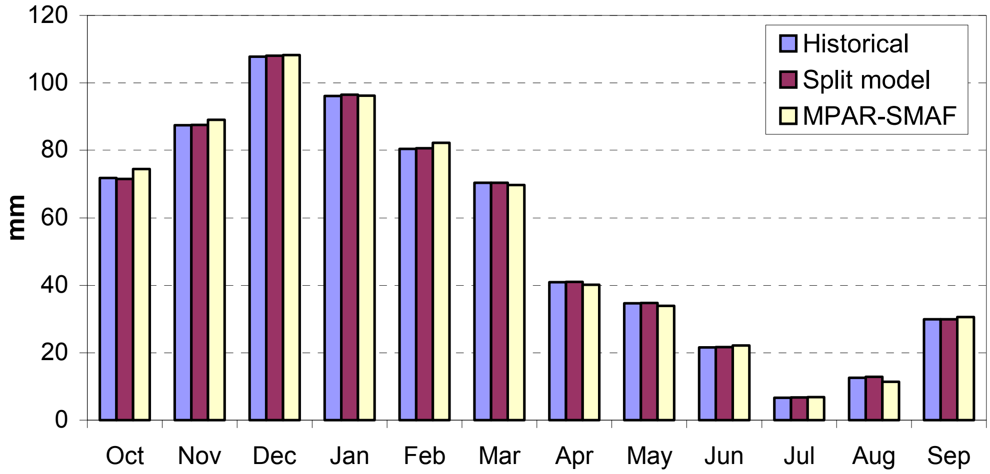

Figure 4 Monthly expected values of the rainfall time series

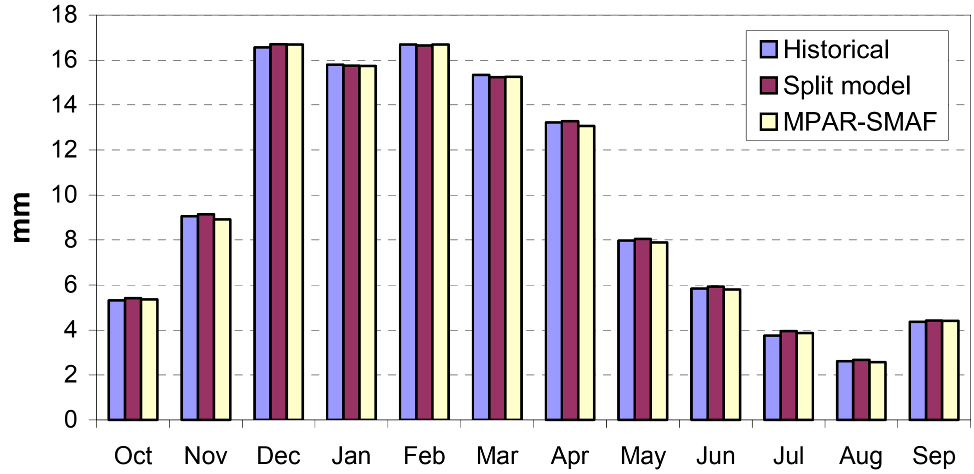

Figure 5 Monthly standard deviations of the discharge time series

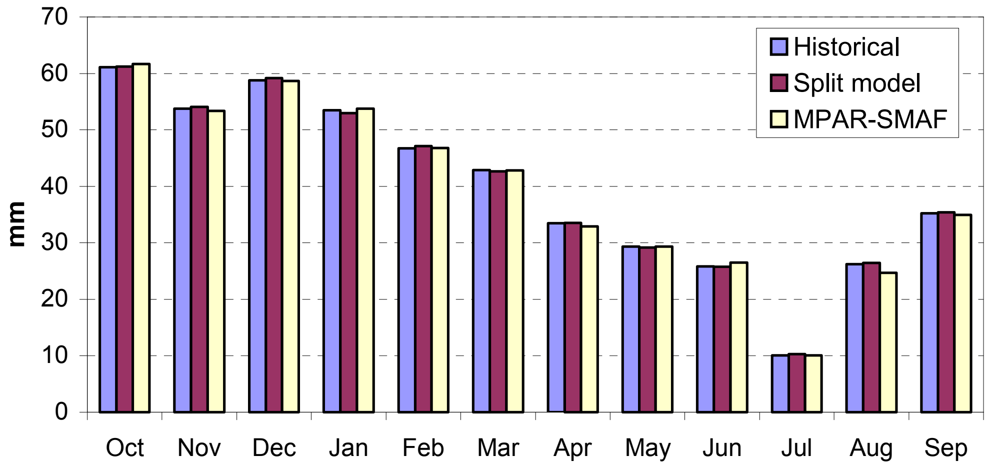

Figure 6 Monthly standard deviations of the rainfall time series

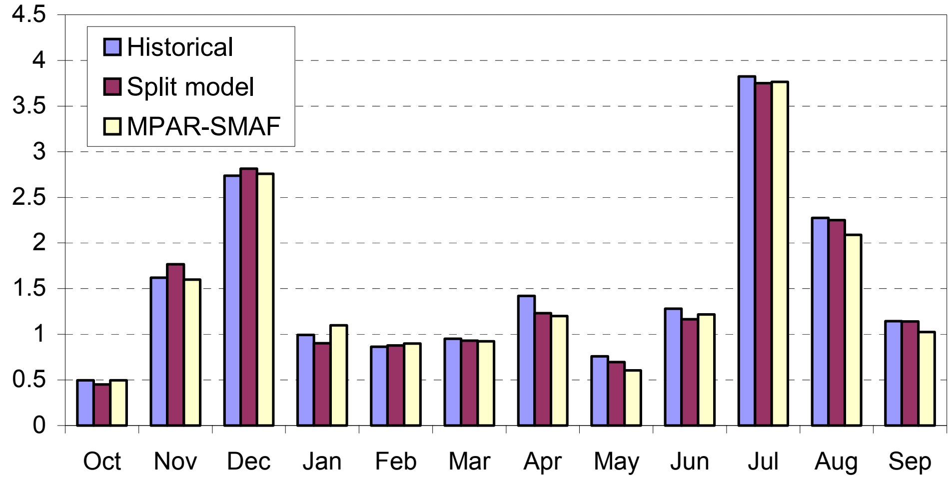

Figure 7 Monthly skewness coefficients of the discharge time series

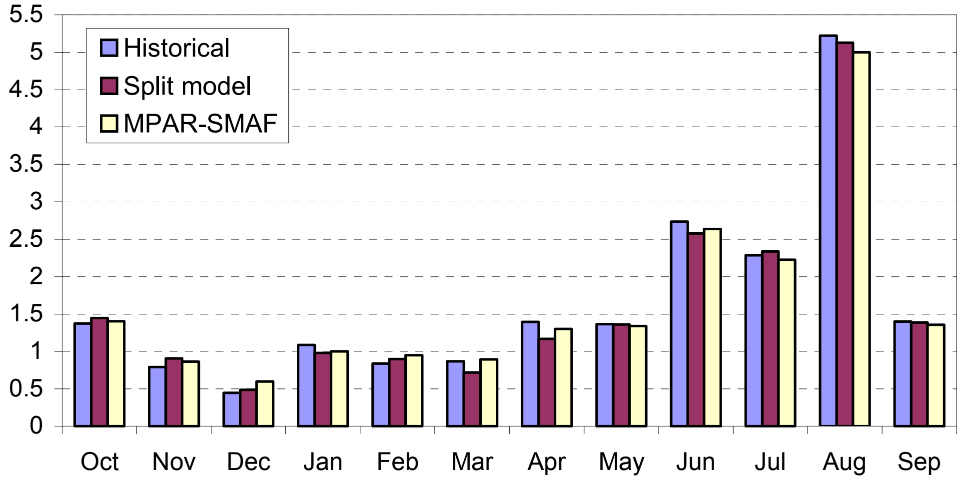

Figure 8 Monthly skewness coefficients of the rainfall time series

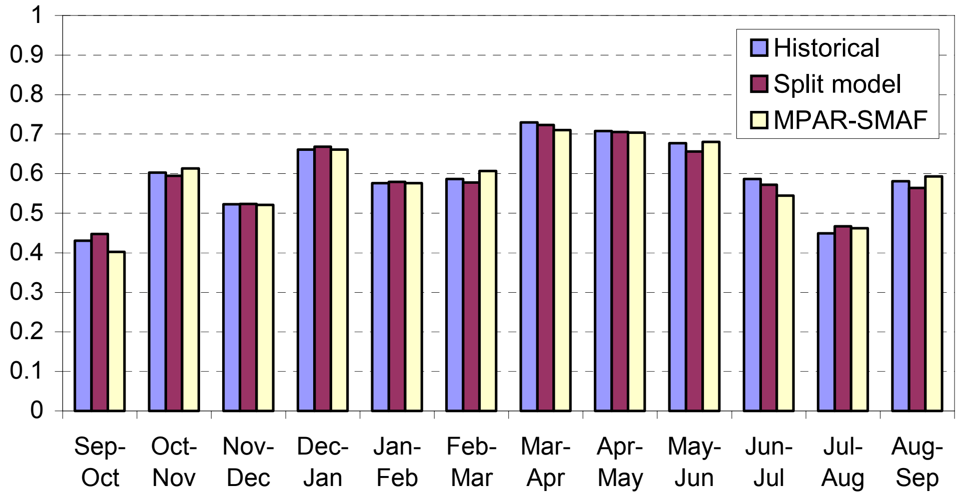

Figure 9 Lag one autocorrelation coefficients of the discharge time series

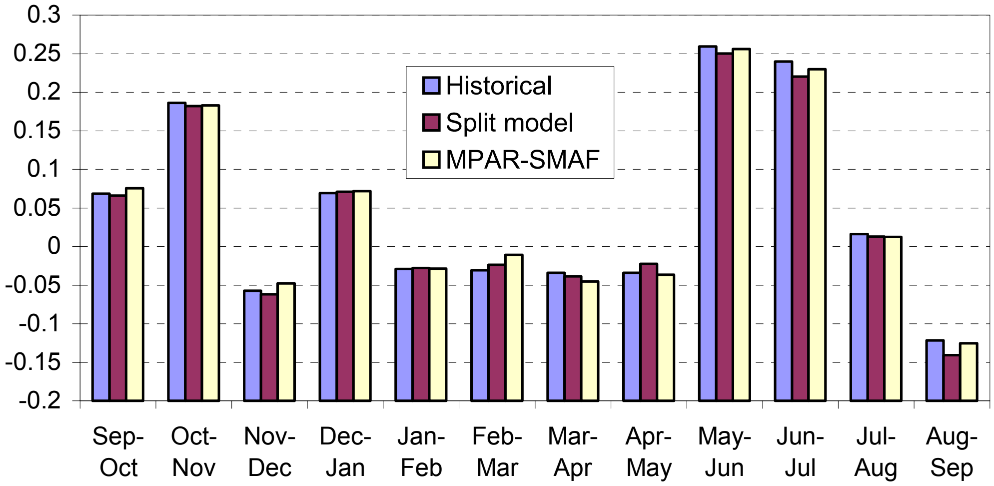

Figure 10 Lag one autocorrelation coefficients of the rainfall time series

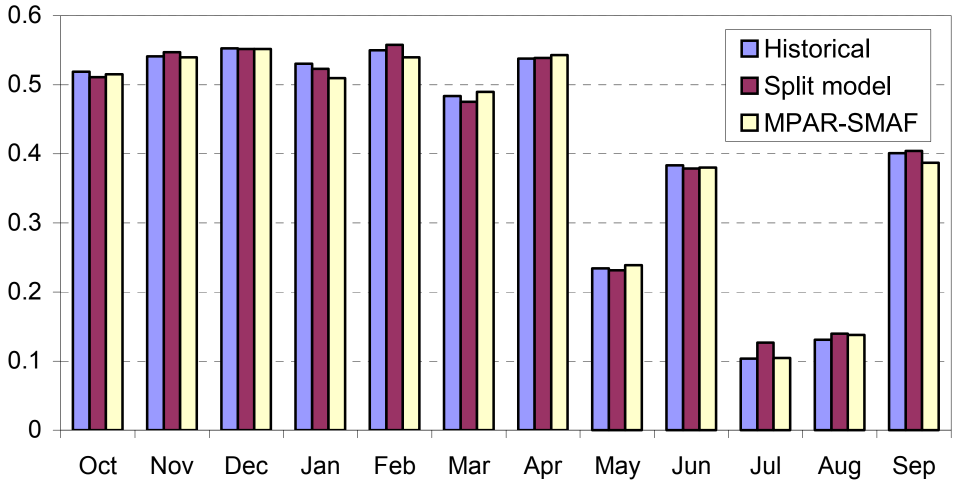

Figure 11 Lag zero cross-correlation coefficients of the two time series

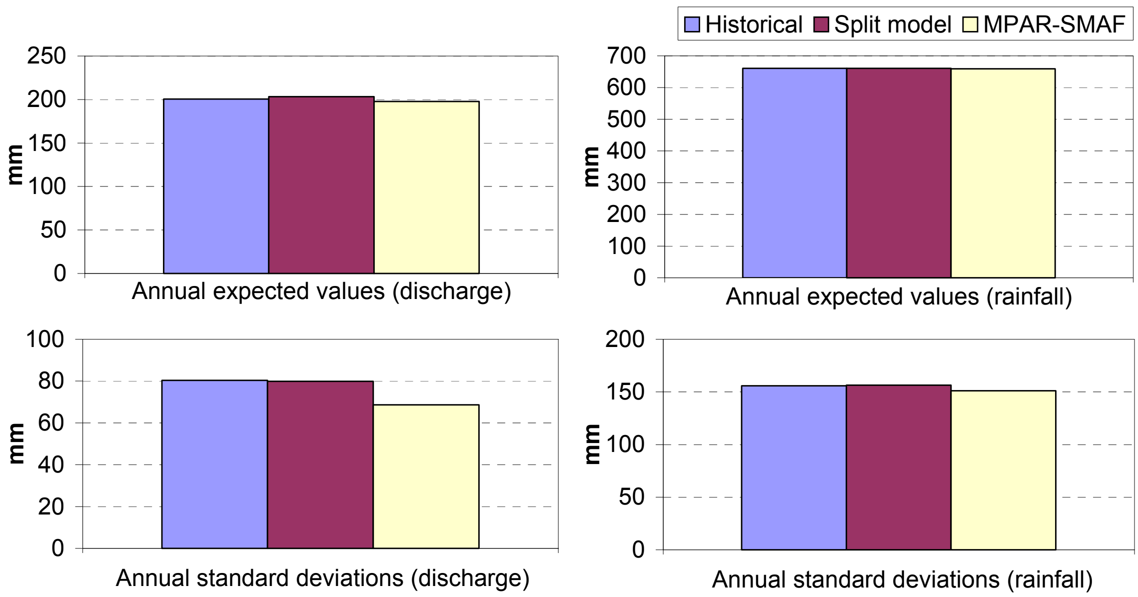

Figure 12 Annual expected values and standard deviations of the discharge and rainfall time series

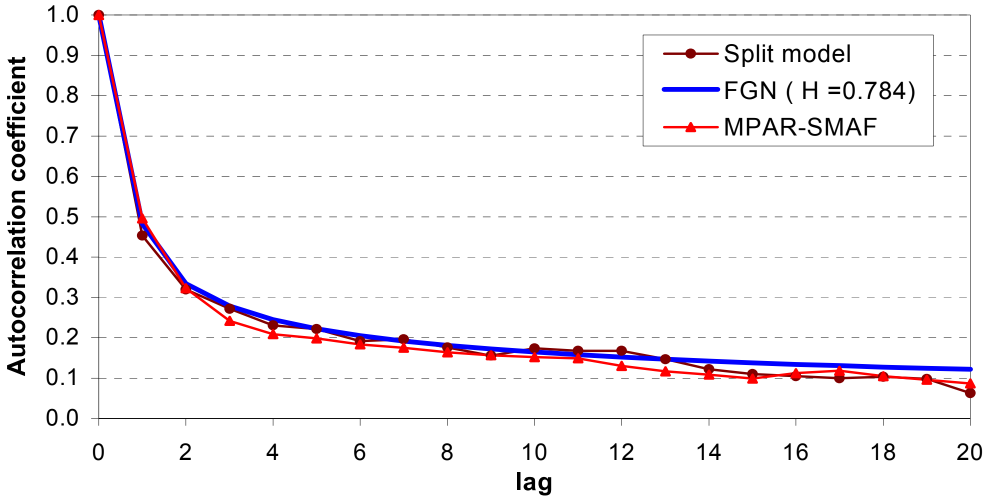

Figure 13 Comparison between the empirical autocorrelogram of the annual discharge time series and the FGN autocorrelogram (Hurst coefficient equal to the one of the historic sample H=0.784)

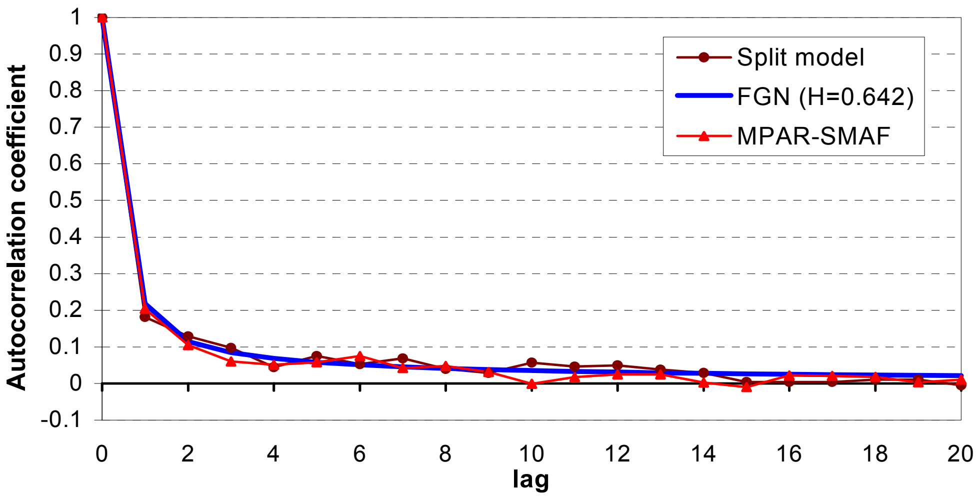

Figure 14 Comparison between the empirical autocorrelogram of the annual rainfall time series and the FGN autocorrelogram (Hurst coefficient equal to the one of the historic sample H=0.642)

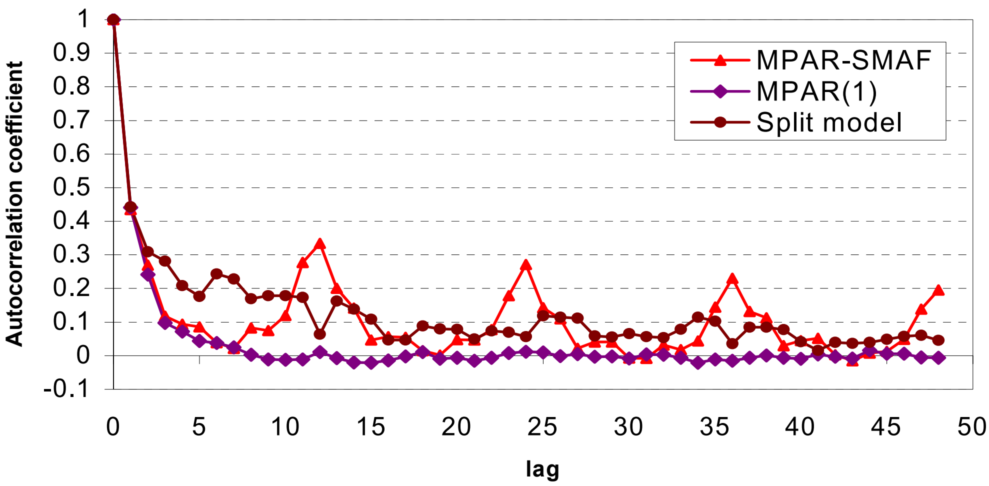

Figure 15 Seasonal autocorrelations of the month of October with previous months, for 3 independent synthetic series produced using MPAR(1), Split and MPAR-SMAF models.

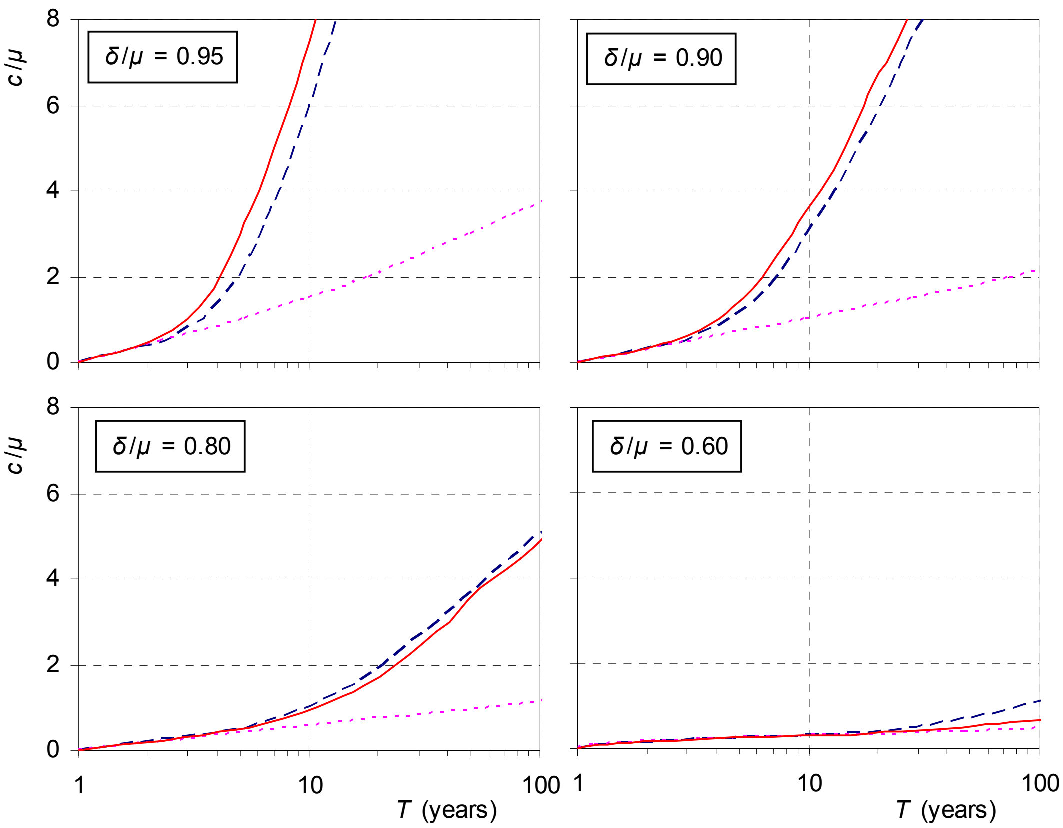

Figure 16 Relationship between recurrence interval of reservoir emptiness (*T*) and reservoir storage capacity standardised by mean inflow (c/μ) for several values of demand standardised by mean (δ/μ), as obtained from simulations using Split model (continuous lines), MPAR-SMAF (dashed lines) and MPAR(1) (dotted lines).