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Long tails of marginal distribution and autocorrelation function of rainfall produced by the maximum entropy principle

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1. Abstract

The long tails of the marginal distribution and the autocorrelation function of rainfall are related to the observed rich patterns in hyetographs, the diversity of rainfall events and even the intermittent behaviour. However, maximization of the classical Boltzmann-Gibbs-Shannon entropy for rainfall at a specific time scale, assuming a specified mean, would result in an exponentially distributed Markovian process. Such a process, with short tails both in the marginal distribution and autocorrelation function, would produce unrealistic rainfall patterns characterized by monotony and without intermittency. Some modified methodologies, which involve the use of a generalized definition of entropy, have been already proposed to reinstate consistency of the maximum entropy principle and observed rainfall behaviour. Here we explore another method which uses the classical entropy definition but assumes that rainfall can be represented as a chain of stochastic processes, each member of which represents the mean of the previous process and has lag one autocorrelation greater than that of the previous process. Application of the method using Monte Carlo simulation demonstrates that such a chain with only three members can produce synthetic traces resembling actual hyetographs.

2. The principle of maximum entropy (ME) and the marginal distribution

- The Boltzmann-Gibbs-Shannon entropy for a continuous random variable X with density function $f(x)$ is by definition (e.g. Shannon and Weaver, 1949; Papoulis, 1991)

$$\varphi = E[-\ln f(x)] = -\int_{-\infty}^{\infty} f(x) \ln f(x) dx$$

- The principle of ME, as formalized by E.T. Jaynes (1957a, b), states that the (unknown) density function $f(x)$ of a random variable X is the one that maximizes the entropy φ, subject to any known constrains.
- Application of the ME principle using the Boltzmann-Gibbs-Shannon entropy with simple constraints of known mean μ and variance σ^2 results in

$$f(x) = \exp(-\lambda_0 - \lambda_1 x - \lambda_2 x^2) \qquad (1)$$

where λ_0, λ_1 and λ_2 are parameters depending on the known mean and variance; inspection of (1) shows that it is the normal density function.
- In statistical physics, if X denotes the momentum of molecules or atoms in a gas volume, the mean and variance constraints correspond precisely to the principles of preservation of momentum and energy.

3. An entropic approach to rainfall – Step 1

- Let X_i denote the rainfall rate at time i discretized at a fine time scale (tending to zero).
- What we definitely know about X_i is $X_i \geq 0$.
- Maximization of entropy with only this condition is not possible.
- Now let us assume that rainfall has a specified mean μ.
- Maximization of entropy with constraints

$$X_i \geq 0, \quad E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \mu$$

results in the exponential distribution: $f(x) = \exp(-x/\mu)/\mu$.
- In addition, let us assume that there is some time dependence of X_i, quantified by $E[X_i X_{i+1}] = \gamma$; this will introduce an additional constraint for the multivariate distribution

$$E[X_i X_{i+1}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i x_{i+1} f(x_i, x_{i+1})\, dx_i dx_{i+1} = \gamma = \rho\, \sigma^2 + \mu^2$$

Here ρ is the correlation coefficient ($\rho > 0$) and σ is the standard deviation (for the exponential distribution $\sigma = \mu$ and thus $\gamma = \rho\, \sigma^2 + \mu^2 = (\rho + 1)\, \mu^2 > \mu^2$).
- Entropy maximization in multivariate setting will result in Markovian dependence.

4. An entropic approach to rainfall – Step 2

- The constant mean constraint in rainfall modelling does not result from a natural principle – as for instance in the physics of an ideal gas, where it represents the preservation of momentum.
- Although it is reasonable to assume a specific mean rainfall, we can allow this to vary in time.
- In this case we can assume that the mean at time i is the realization of a random process M_i which has mean μ and lag 1 autocorrelation $\rho^M > \rho$.
- Application of the ME principle will produce that M_i is Markovian with exponential distribution.
- Then application of conditional distribution algebra results in

$$f(x) = 2\, K_0(2\, (x/\mu)^{1/2})/\mu, \quad F(x) = 1 - 2\, (x/\mu)^{1/2}\, K_1(2\, (x/\mu)^{1/2})/\mu$$

where $K_n(x)$ is the modified Bessel function of the second kind (important observation: $f(0) = \infty$, whereas in the exponential distribution $f(0) = \mu < \infty$).
- The moments of this distribution are $E[X^n] = \mu^n\, n!^2$ (note: in exponential distribution $E[X^n] = \mu^n\, n!$) so that

$$E[X] = \mu, \quad \mathrm{Var}[X] = 3\, \mu^2 \;\rightarrow\; C_V = \sigma/\mu = \sqrt{3} > 1$$

- The dependence structure becomes more complex than Markovian (difficult to find an analytical solution).

5. An entropic approach to rainfall – Step 3

- Proceeding in a similar manner as in step 2, we can now replace the constant mean μ of the process M_i with a varying mean, represented by another stochastic process N_i with mean μ and lag 1 autocorrelation $\rho^N > \rho^M > \rho$.
- In this manner we can construct a chain of processes, each member of which represents the mean of the previous process.
- By construction, the lag 1 autocorrelations of these processes form a monotonically increasing sequence, i.e. …. $> \rho^N > \rho^M > \rho$.
- The scale of change or fluctuation of each process of the chain is a monotonically increasing sequence, i.e. …. $> q^N > q^M > q$, where $q := (-\ln \rho)^{-1}$; the scale of fluctuation represents the time required for the process to decorrelate down to an autocorrelation 1/e.
- The (unconditional) mean of all processes is the same, μ.
- All moments except the first form an increasing sequence as we proceed through the chain; higher moments increase more.
- Analytical handling of the marginal distribution and the dependence structure is very difficult.
- However we can easily inspect the idea using Monte Carlo simulation.

6. A demonstration using a chain with 3 processes

- Simulation of a Markovian process with exponential distribution is easy and precise; there are several methodologies to implement it.
- Here we implement an Exponential Markov (EM) process as

$$X_i = \mu\, [-\ln G(Y_i)]$$

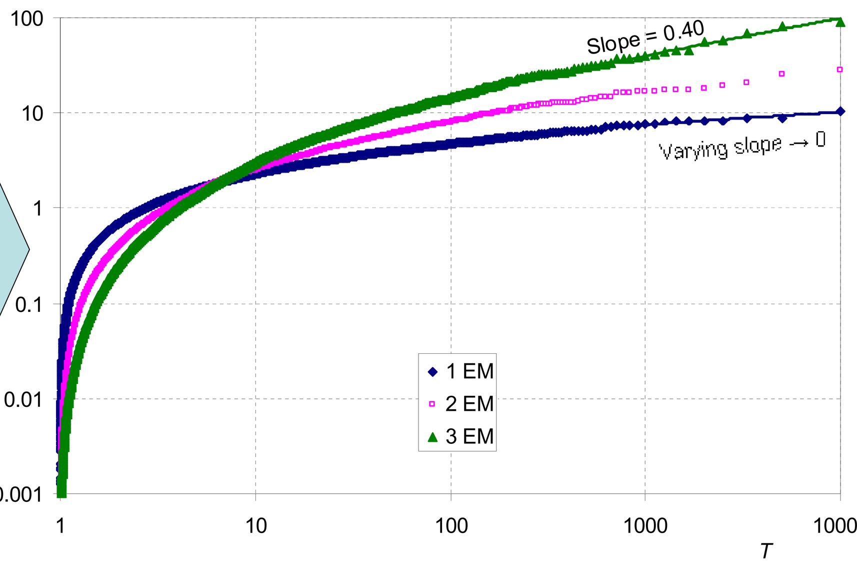
where μ is the mean, Y_i is a standard AR(1) process with standard normal distribution and G( ) is the standard normal distribution function.
- Simulations with a length 10 000 were performed for the following cases (for comparison).

	Case	1 EM	2 EM		3 EM		
	Process	X	M	X	N	M	X
Processes in chain	Mean	1	1	-	1	-	-
	Lag 1 autocorrelation*	0.48	0.9	0.25	0.99	0.85	0.2
	Scale of fluctuation	1.37	9.5	0.72	99.5	6.2	0.62
Final process (X)	Mean	1	1		1		
	Standard deviation	1	1.73		3.30		
	Lag 1 autocorrelation	0.48	0.48		0.48		

* Autocorrelation coefficients refer to the standard AR(1) process but are approximately equal in the EM process.

7. Simulation results – distribution function

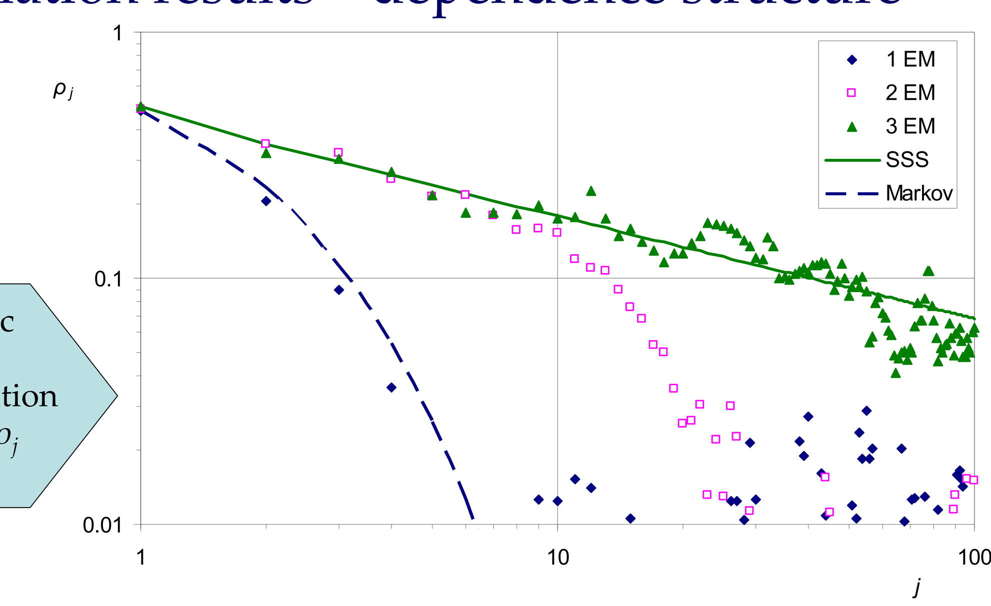
Logarithmic plot of "rainfall intensity" (x) vs. empirically estimated return period ($T(x) := 1/[1 - F(x)]$, where $F(x)$ is the distribution function)



As the number of processes in the chain increases, the right tail of the distribution moves toward higher "rainfall intensity" values and its shape changes from exponential type to power type; simultaneously the probability density becomes infinite for $x = 0$.

8. Simulation results – dependence structure

Logarithmic plot of autocorrelation coefficient ρ_j vs. lag j

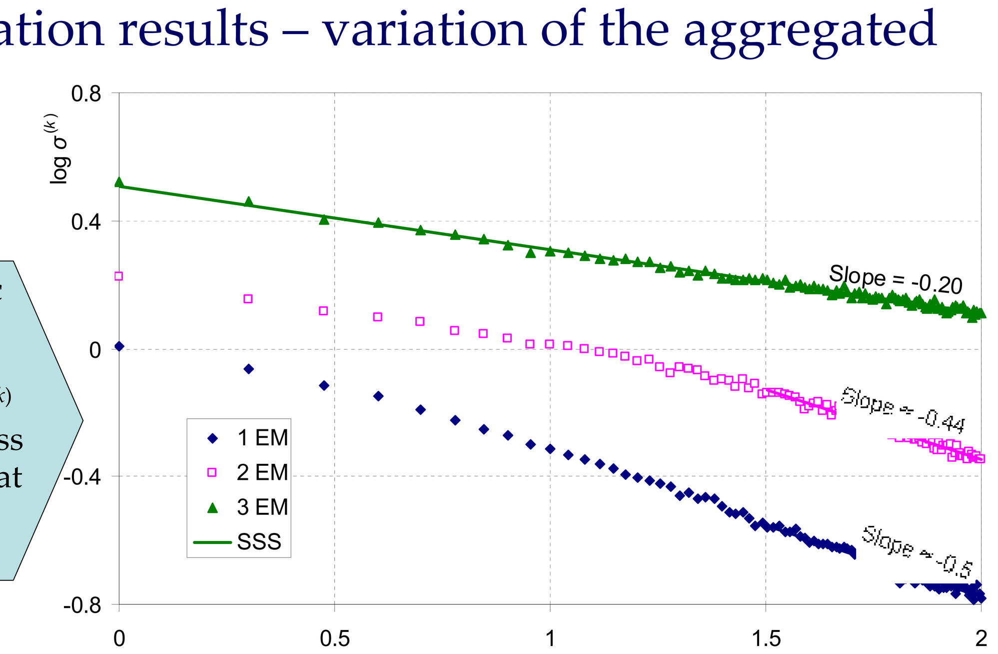


As the number of processes in the chain increases, the shape of the autocorrelation function changes from Markovian (exponential decay – short range dependence) to power type (long range dependence).

The latter type is characteristic of the Hurst-Kolmogorov behaviour, which can be represented by a simple scaling stochastic process (SSS process).

9. Simulation results – variation of the aggregated process

Logarithmic plot of standard deviation $\sigma^{(k)}$ of the process aggregated at scale k, vs. scale k



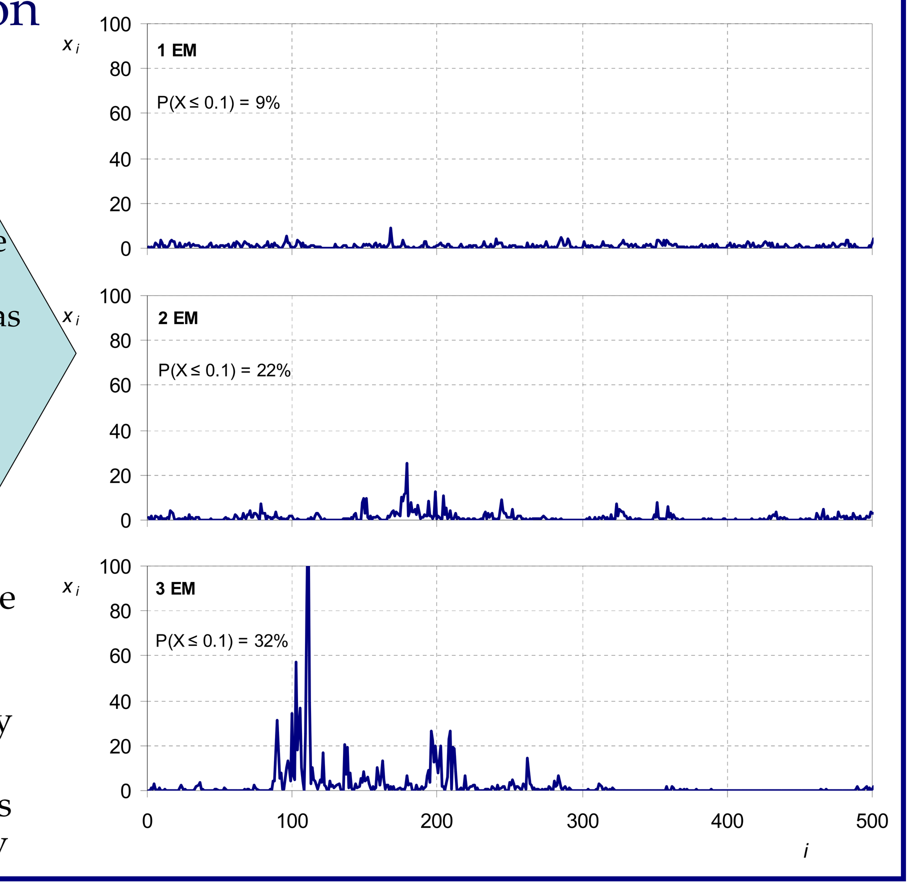
The slope of the logarithmic plot (as $k \to \infty$) is $H - 1$ where H is the Hurst exponent.

The slope in the "1 EM" case is –0.5, i.e. $H = 0.5$, meaning no Hurst-Kolmogorov behaviour.

The slope in "3 EM" is –0.20, i.e. $H = 0.80$, suggesting a Hurst-Kolmogorov behaviour.

10. Simulation results – general behaviour

Plots of parts of the generated time series (selected so as to include the maximum over 10 000 generated values)



As the number of processes in the chain increases the general shape changes:

From monotony to rich patterns

From steadiness to intermittency

11. Can entropy maximization be performed in a single step? (The Tsallis entropy)

- A generalization of the Boltzmann-Gibbs-Shannon entropy has been proposed by Tsallis (1998, 2004)

$$\varphi_q = \frac{1 - \int_{-\infty}^{\infty} [f(x)]^q}{q - 1}$$

with $q = 1$ corresponding to the Boltzmann-Gibbs-Shannon entropy.
- Maximization of Tsallis entropy with known μ yields

$$f(x) = [1 + \kappa\, (\lambda_0 + \lambda_1\, x)]^{-1 - 1/\kappa}, \quad x \geq 0$$

where $\kappa := (1 - q)/q$ and λ_0, λ_1, λ_2 and are parameters.
- Clearly, this is the Pareto distribution and has an over-exponential (power-type) distribution tail.
- Whilst this approach succeeds in producing a long tail to the right, it fails in reproducing the tail to the left (it underpredicts the probability of very low values; see Papalexiou and Koutsoyiannis, 2008).
- Furthermore, a single-step approach based on the Tsallis entropy cannot reproduce the Hurst-Kolmogorov behaviour; to remedy this, a multi-scale setting of the entropy maximization has been proposed by Koutsoyiannis (2005).

12. Conclusions

- The principle of maximum entropy provides a sound theoretical basis for studying the rainfall process.
- However, maximization of the classical Boltzmann-Gibbs-Shannon entropy at a specific time scale, assuming a specified mean, would result in an exponentially distributed Markovian process, which is unrealistic.
- This can be remedied by performing the entropy maximization in several steps, thus representing rainfall as a chain of stochastic processes, each member of which represents the mean of the previous process and has lag one autocorrelation greater than that of the previous process.
- Monte Carlo simulation demonstrates that such a chain with only three members can produce synthetic traces resembling actual hyetographs.
- The resulting model is characterized by high autocorrelation at fine scales, slowly decreasing with lag (Hurst-Kolmogorov behaviour), by long distribution tails, and by probability density tending to infinity for rainfall intensity tending to zero.

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