8th IAHS Scientific Assembly
& 37th IAH Congress

6 - 12 September 2009, Hyderabad, India

IAHS Workshops Session HW7
New statistics in Hydrology

Ombrian curves: from theoretical consistency to engineering practice

S.M. Papalexiou and D. Koutsoyiannis
Department of Water Resources and Environmental Engineering, National Technical University of Athens, Greece (www.itia.ntua.gr)

1. Abstract

One of the major tools in hydrological design is the ombrian curves, more widely known by the misnomer rainfall intensity-duration-frequency (IDF) curves. An ombrian curve is a mathematical relationship estimating the average rainfall intensity over a given timescale for a given return period. Several forms of ombrian curves are found in the literature, most of which have been empirically derived and validated by the long use in hydrological practice. Attempts to give them a theoretical basis have often used inappropriate assumptions (e.g. simple scaling) and resulted in oversimplified relationships that are not good for engineering studies. In a previous study, we have derived theoretically consistent ombrian curves based on a probability distribution suitable for describing the average rainfall intensity over a wide range of timescales (from sub-hourly to yearly). The mathematical form of those theoretically derived ombrian curves is not as simple as other widely used forms in hydrological practice. In this study, we present simplified ombrian relationships, which are approximations of the theoretically consistent one for a typical range of timescales, suitable for use in hydrological engineering.

2. Approximation 1 (more accurate)

- In a previous study (Papalexiou and Koutsoyiannis, 2008), we suggested that the expression of theoretically consistent ombrian curves should be of the form $i(k, T) = Q\,(1 - (k/T)(1/p_w(k)); \boldsymbol{\theta}(k))$, where k is the timescale, T is the return period, Q is the quantile function of a 4-parameter probability distribution (the JH distribution) capable of describing the rainfall intensity i in a wide range of timescales, $p_w(k)$ is the probability wet for timescale k, and $\boldsymbol{\theta}(k)$ is a vector of the distribution parameters (Note: the expression assumes consistent units).
- In this study, in order to simplify the procedure and to construct a consistent approximation, we use a three-parameter version of the JH distribution, known in the literature as the Burr type VII distribution. Its quantile function is $Q(u)= a[(1 - u)^{-1/c} - 1]^{1/b}$.
- The proposed methodology for constructing approximations of ombrian curves includes:

1. **Estimation of the tail exponent.** It can be proved mathematically, that the exponent of the power-type decay of the exceedence probability function of the average rainfall intensity i is the same for all timescales. For the Burr type VII distribution this implies that the product $b(k)\, c(k) =: 1/\beta$ is constant for all scales.
2. **Estimation of probability wet $p_w(k)$.** A suitable function for the probability wet in all timescales may be found in the general family of functions $p_w(k) = (1- p_w(0)\,)\, f_1(k) + p_w(0)$, where $p_w(0)$ is the probability wet in continuous time, and $f_1(k)$ is any monotonically increasing function in $[0,\infty)$ with range [0,1].
3. **Estimation of the first and second raw moments $m_1(k)$ and $m_2(k)$.** The first raw moment $m_1(k)$, or the mean, of the rainfall intensity of nonzero rainfall periods is explicitly related to the probability wet, i.e., $m_1(k) = m/p_w(k)$, where m is the mean value of the rainfall intensity including zeros, constant over all timescales. For the second raw moment, an appropriate function may be found in the family $m_2(k) = \{f_2(k) + [m_2(0) - m^2]^{-1}\}^{-1} + m^2$, where $m_2(0)$ is the second raw moment in continuous time and $f_2(k)$ is any monotonically increasing function in $[0,\infty)$ with range $[0,\infty]$.
4. **Estimation of the distribution's parameter functions $a(k)$ and $b(k)$.** The parameters a and b at timescale k can be estimated by solving the system that results by equating the first two raw moments of the theoretical distribution with the respective numerical estimates resulting from the fitted laws in step 3. If the parameters a and b are estimated in a sufficiently large number of timescales, the interpolation function $a(k)$ and $b(k)$ may be constructed.

3. Approximation 2 (more parsimonious)

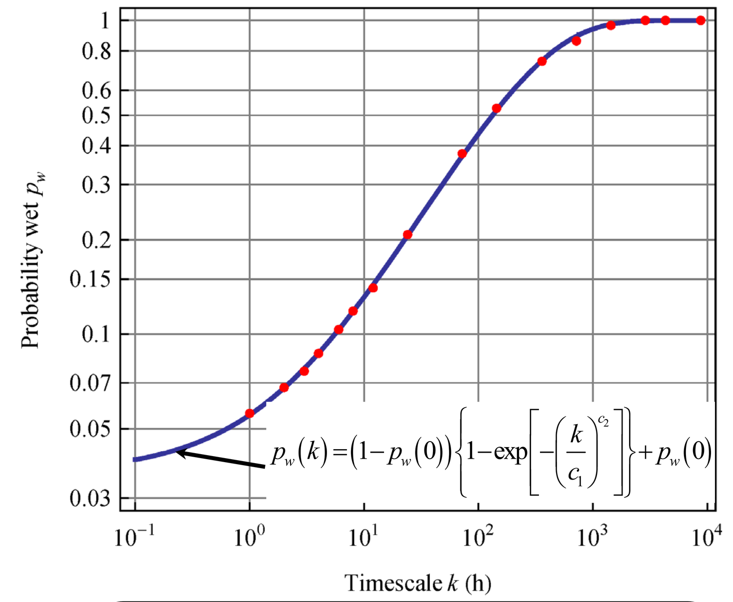
- Substitution in the theoretically consistent ombrian relationship of the quantile function of the Burr VII distribution results in
$$i(k, T) = a(k)\,\{[(k/T)(1/p_w(k))]^{-b(k)\,\beta} - 1\}^{1/b(k)} = a(k)\,[g(k)\,T^{\,b(k)\,\beta} - 1]^{1/b(k)}$$
where $g(k) := [p_w(k)/k]^{b(k)\,\beta}$.
- Numerical investigation shows that b(k) can be cancelled out in the term $[g(k)\,T^{\,b(k)\,\beta} - 1]^{1/b(k)}$ and thus this term can be well approximated as $g'(k)\,(T^\beta - \zeta)$, so that
$$i(k, T) = a'(k)\,(T^\beta - \zeta)$$
where $a'(k) := a(k)\,g'(k)$ and ζ is a constant.
- Numerical investigation shows that the function $a'(k)$ can be approximated as $a'(k) = (k + \delta)^\varepsilon/\alpha$, where α, δ and ε are constants.
- This leads to the simple ombrian expression
$$i(k, T) = (\alpha\, T^\beta - \gamma) \,/\, (k + \delta)^\varepsilon$$
- The above derivation would also be obtained by assuming a generalized Pareto distribution for i and functional separability of return period T and timescale k (see Koutsoyiannis *et al.*, 1998, Koutsoyiannis and Baloutsos, 2000).
- The final form resembles the expressions used in practice except for the constant term γ in the numerator.

4. Probability wet vs. timescale (NOA Greece)

Summary statistics of the average rainfall intensity data (mm h^{-1}), for several different timescales ranging from 1 h to 1 year for the NOA station.

Scale	[a]p_w	Mean	[b]SD	[c]C_V	[d]C_S	Max
1 h	0.06	0.70	1.71	2.43	7.36	58.56
2 h	0.07	0.58	1.30	2.24	6.53	38.25
3 h	0.08	0.51	1.08	2.10	6.34	33.98
4 h	0.09	0.45	0.92	2.05	5.88	26.71
6 h	0.10	0.38	0.74	1.96	5.47	19.25
8 h	0.12	0.33	0.62	1.89	4.73	13.36
12 h	0.14	0.28	0.50	1.80	5.25	12.15
24 h	0.21	0.18	0.31	1.69	4.24	6.08
3 d	0.38	0.10	0.15	1.46	3.43	2.03
6 d	0.53	0.07	0.09	1.31	2.87	1.02
15 d	0.74	0.05	0.06	1.13	2.52	0.65
30 d	0.86	0.04	0.04	1.00	1.66	0.34
60 d	0.96	0.04	0.03	0.92	1.49	0.25
120 d	1.00	0.04	0.03	0.76	0.85	0.14
180 d	1.00	0.04	0.02	0.57	0.45	0.10
365 d	1.00	0.04	0.01	0.28	0.21	0.07

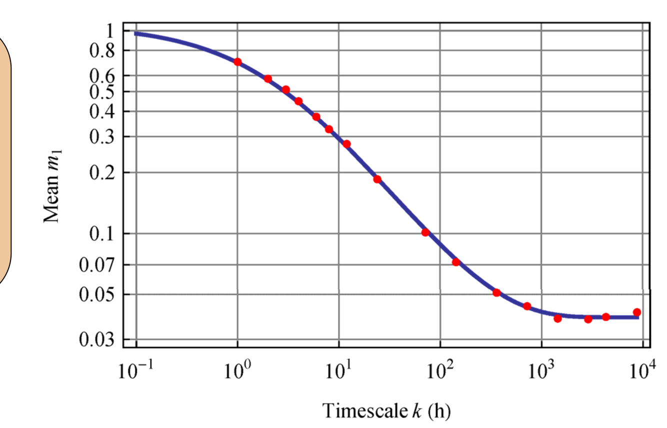
[a]Probability wet [b]Standard deviation
[c]Coefficient of variation [d]Coefficient of skewness



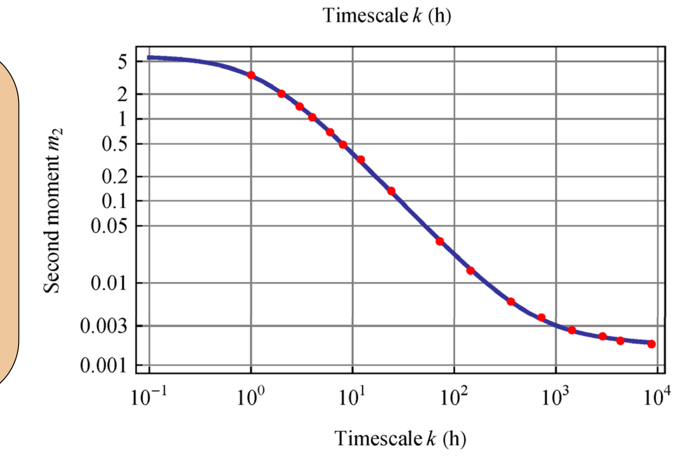
The figure depicts the empirical probability wet (red dots) for several different timescales. Additionally, a theoretical expression was fitted to the empirical data (blue solid line), with estimated parameters: $p_w(0) = 0.04$, $c_1 = 240.74$ h, $c_2 = 0.71$, which permits extrapolation for timescales smaller than that of observation down to 0 (whereas for very large timescales $p_w(k) = 1$).

5. Moments vs. timescale (NOA Greece)

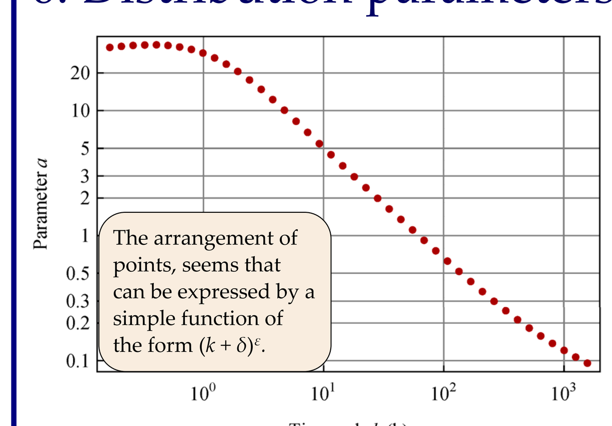
The mean m_1 of the nonzero rainfall intensity in timescale k is explicitly related to the probability wet. In the figure the red dots represent the sample mean m_1, at various timescales, for the NOA station, while the solid blue line depicts the theoretical expression $m_1(k) = m/p_w(k)$.

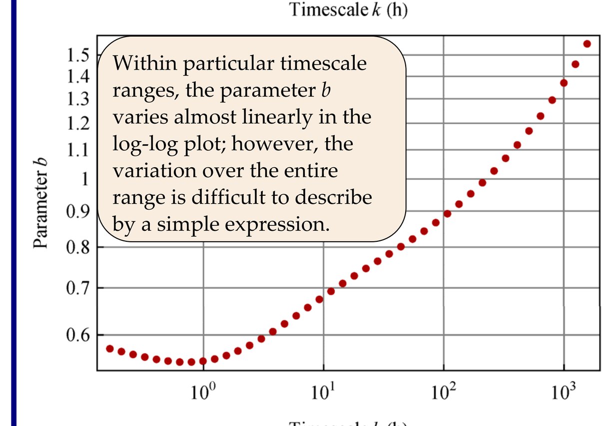


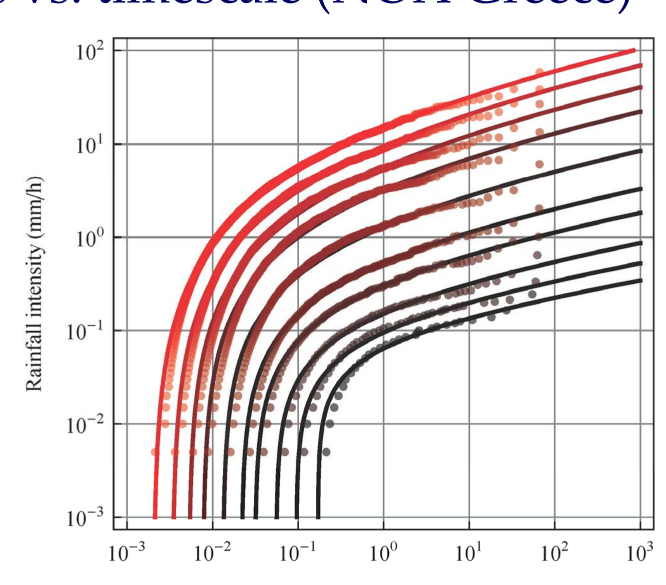
The figure depicts the sample estimates of the second raw moment m_2 of the rainfall intensity in various timescales (red dots) for the NOA station, and a fitted theoretical expression (blue solid line) of the form $m_2(k) = 1/\{c_4 \ln[(k/c_5)^{c_6} + 1] + 1/[m_2(0) - m^2]\} + m^2$ The estimated parameters are $m_2(0) = 5.80$ mm^{2}h^{-2}, $c_4 = 858.41$ mm^{-2}h^2, $c_5 = 915.05$ h, $c_6 = 1.29$.



6. Distribution parameters vs. timescale (NOA Greece)

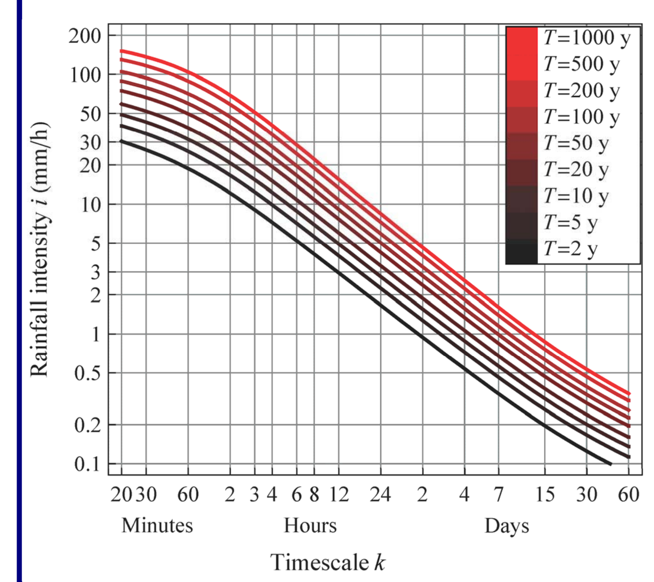




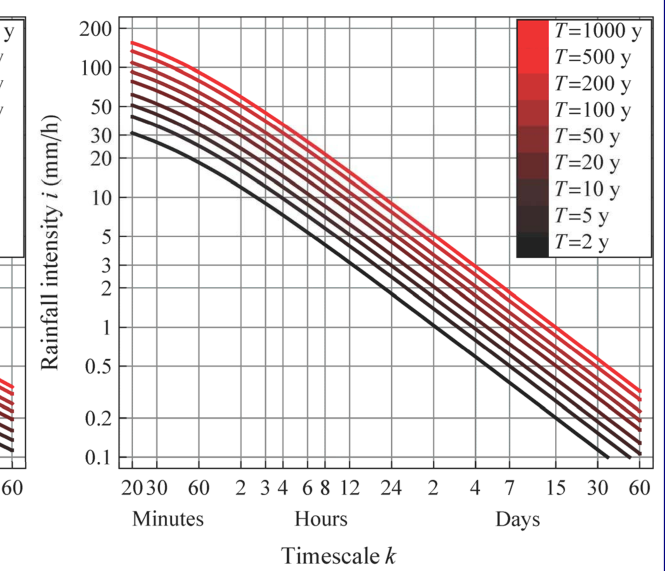


Empirical distributions of the aggregated rainfall intensity for the NOA station over several timescales, and the Burr type VII distributions fitted by using parameter values the ones resulted by the interpolation functions $a(k)$ and $b(k)$, with $a(k)\,b(k) = 6.5$ (= 1/0.15). Timescales from above to below: 1 h, 2 h, 3 h, 4 h, 8 h, 24 h, 3 d, 6 d, 15 d, 30 d, 60 d.

7. Ombrian curves (NOA Greece)



Ombrian curves constructed for the NOA station using Approximation 1.

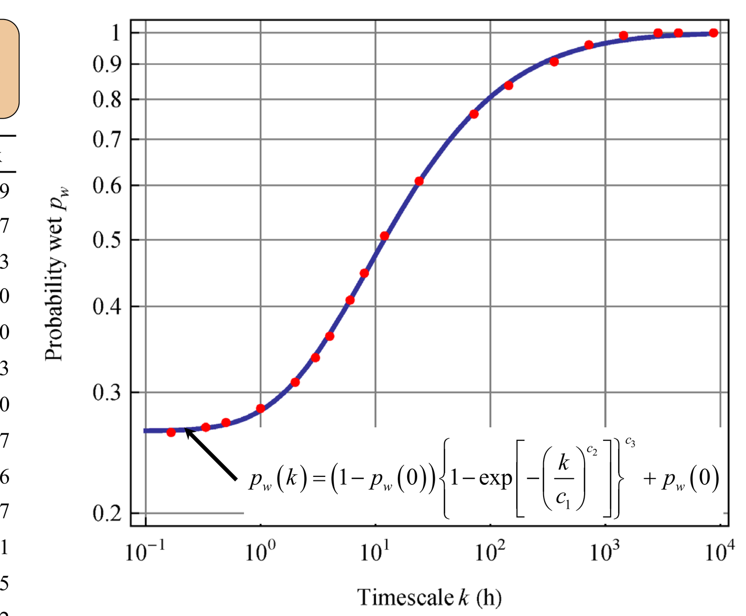


Ombrian curves constructed for the NOA station using Approximation 2. The estimated parameters are: $\alpha = 41.16$, $\beta = 0.18$, $\gamma = 22.15$, $\delta = 0.41$, $\varepsilon = 0.82$.

8. Probability wet vs. timescale (Ardeemore UK)

Summary statistics of the average rainfall intensity (mm h^{-1}), for several different timescales for the Ardeemore station.

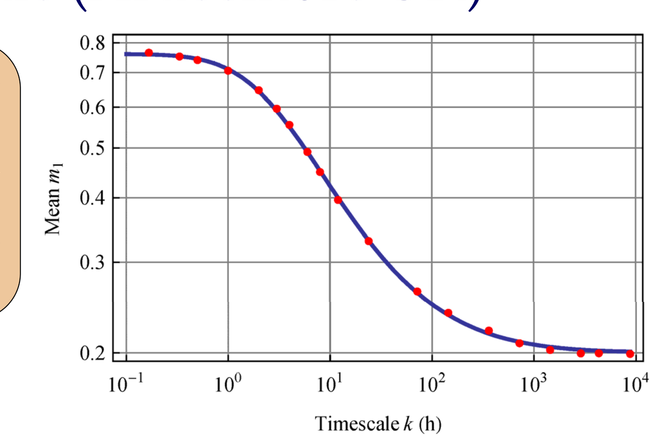
Scale	p_w	Mean	SD	C_V	C_S	Max
10 min	0.20	0.98	1.47	1.49	6.16	56.49
20 min	0.21	0.95	1.31	1.38	5.08	52.87
30 min	0.21	0.92	1.21	1.31	4.19	35.43
1 h	0.23	0.84	1.04	1.23	3.37	20.00
2 h	0.27	0.72	0.87	1.20	3.05	17.60
3 h	0.31	0.64	0.77	1.20	2.79	12.93
4 h	0.34	0.58	0.71	1.21	2.67	9.70
6 h	0.39	0.51	0.60	1.19	2.46	6.47
8 h	0.43	0.46	0.54	1.18	2.32	5.46
12 h	0.49	0.40	0.46	1.14	2.22	4.07
24 h	0.60	0.33	0.35	1.06	1.85	2.61
3 d	0.75	0.26	0.24	0.93	1.70	1.75
6 d	0.83	0.24	0.20	0.85	1.78	1.62
15 d	0.91	0.22	0.17	0.76	2.33	1.45
30 d	0.96	0.20	0.13	0.66	1.62	0.93
60 d	0.99	0.20	0.12	0.62	1.91	0.93
120 d	1.00	0.20	0.09	0.47	1.10	0.59
180 d	1.00	0.20	0.08	0.41	0.32	0.45
365 d	1.00	0.19	0.06	0.33	-0.63	0.30



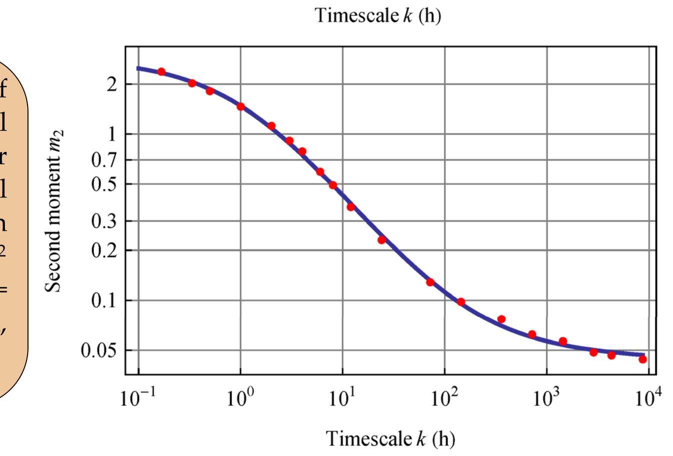
The figure depicts the empirical probability wet (red dots) for several different timescales. Additionally, a theoretical expression was fitted to the empirical data (blue solid line), with estimated parameters: $p_w(0) = 0.26$ h, $c_1 = 0.00002$, $c_2 = 0.12$, $c_3 = 111.77$.

9. Moments vs. timescale (Ardeemore UK)

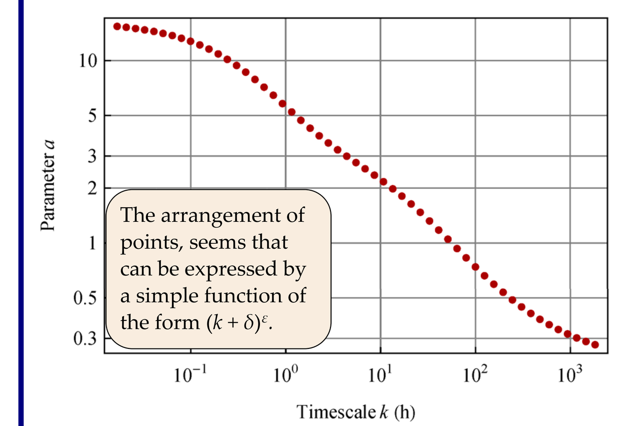
The mean m_1 of the nonzero rainfall intensity in timescale k is explicitly related to the probability wet. In the figure the red dots represent the sample mean m_1, at various timescales, for the Ardeemore station, while the solid blue line depicts the theoretical expression $m_1(k) = m/p_w(k)$.

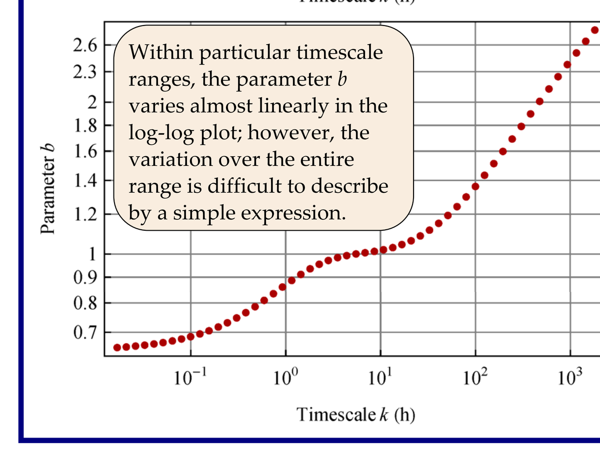


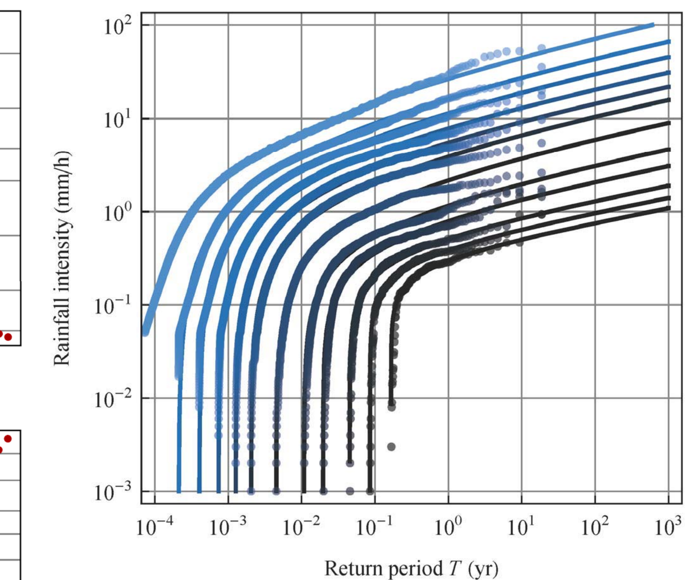
The figure depicts the sample estimates of the second raw moment m_2 of the rainfall intensity in various timescales (red dots) for the NOA station, and a fitted theoretical expression (blue solid line) of the form $m_2(k) = 1/\{c_4 \ln[(k/c_5)^{c_6} + 1] + 1/[m_2(0) - m^2]\} + m^2$ The estimated parameters are $m_2(0) = 2.83$ mm^{2}h^{-2}, $c_4 = 61.98$ mm^{-2}h^2, $c_5 = 546.33$ h, $c_6 = 0.83$.



10. Distribution's parameters vs. timescale (Ardeemore UK)

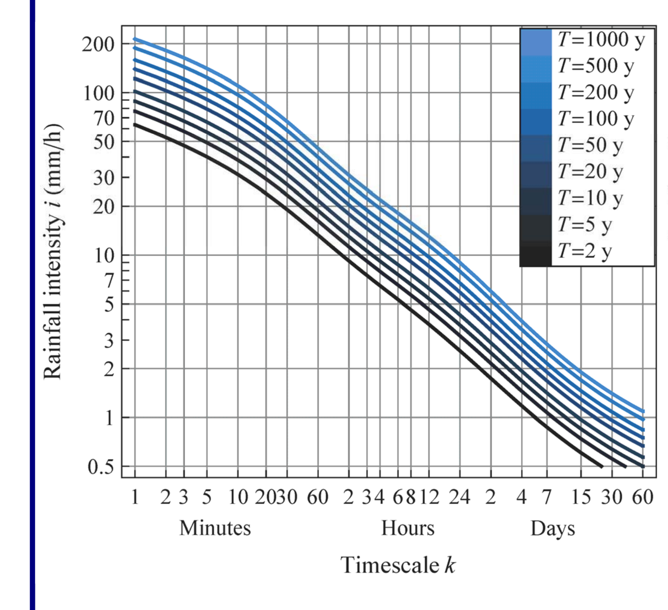




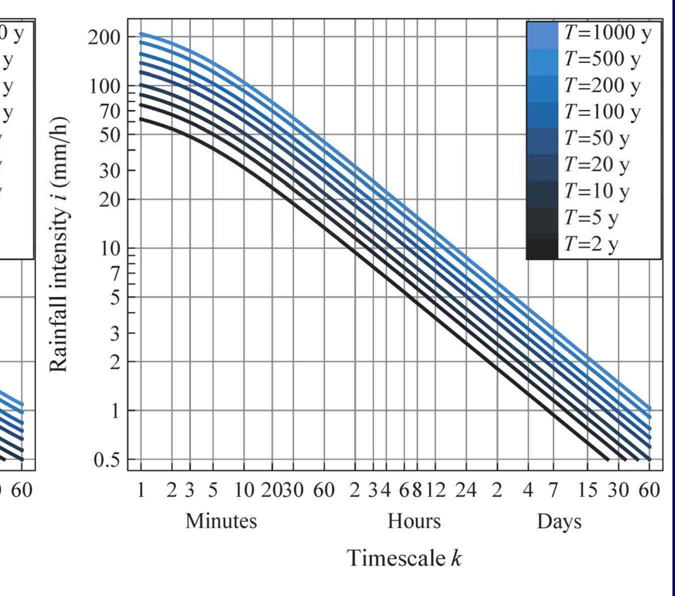


Empirical distributions of the aggregated rainfall intensity for the Ardeemore station over several timescales, and the Burr type VII distributions fitted by using parameter values the ones resulted by the interpolation functions $a(k)$ and $b(k)$, with $a(k)\,b(k) = 6.5$ (= 1/0.15). Timescales from above to below: 10 min, 30 min, 1 h, 2 h, 3 h, 4 h, 8 h, 24 h, 3 d, 6 d, 15 d, 30 d, 60 d.

11. Ombrian curves (Ardeemore UK)



Ombrian curves constructed for the Ardeemore station using Approximation 1.



Ombrian curves constructed for the Ardeemore station using Approximation 2. The estimated parameters are: $\alpha = 18.44$, $\beta = 0.15$, $\gamma = 6.87$, $\delta = 0.04$, $\varepsilon = 0.52$.

12. Conclusions

- Ombrian curves are expressions linking the probability distribution of average rainfall intensity i with the timescale k on which the average is taken.
- Thus a theoretically consistent expression of ombrian curves is non other than the expression of the multi-scale probability distribution of average rainfall intensity.
- It has been demonstrated (Papalexiou and Koutsoyiannis, 2009) that a 4-parameter distribution can accurately represent this multi-scale distribution for scales ranging from sub-hourly to yearly.
- Two approximations of this expression are developed that may facilitate a consistent development of ombrian curves for engineering practice.
- The first is $i(k, T) = a(k)\,[g(k)\,T^{\,b(k)\,\beta} - 1]^{1/b(k)}$ where the functions $a(k)$, $b(k)$ and $g(k)$ can be estimated by a simple but laborious methodology.
- The second is a simplification of the first one and implies the functional separability of return period T and timescale k in the form $i(k, T) = (\alpha\, T^\beta - \gamma) \,/\, (k + \delta)^\varepsilon$.

References

- Koutsoyiannis, D., and G. Baloutsos, Analysis of a long record of annual maximum rainfall in Athens, Greece, and design rainfall inferences, *Natural Hazards*, 22 (1), 31–51, 2000.
- Koutsoyiannis, D., D. Kozonis, and A. Manetas, A mathematical framework for studying rainfall intensity-duration-frequency relationships, *Journal of Hydrology*, 206 (1-2), 118–135, 1998.
- Papalexiou, S.M., and D. Koutsoyiannis, Probabilistic description of rainfall intensity at multiple time scales, *IHP 2008 Capri Symposium: "The Role of Hydrology in Water Resources Management"*, Capri, Italy, UNESCO, International Association of Hydrological Sciences, 2008.
- Papalexiou, S.M., and D. Koutsoyiannis, An all-timescales rainfall probability distribution, *European Geosciences Union General Assembly 2009, Geophysical Research Abstracts, Vol. 11*, Vienna, 13469, European Geosciences Union, 2009.